شبه قابلية الانعكاس للمصفوفات فوق شبه الحلقات التبديلية

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الملخص

هذه الورقة تبحث في العلاقة بين مفهومي قابلية الانعكاس و شبه قابلية الانعكاس للمصفوفات المربعة فوق شبه الحلقات التبديلية ، متضمنة الروابط بين أساس الفضاء شبه الخطي و شبه قابلية الانعكاس. الشرط الكافي للمصفوفات المربعة فوق شبه الحلقات التبديلية لأن تكون شبه قابلة للانعكاس يكون متحصل عليه.

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Abstract :

This paper deals with the relationship between the concepts of invertibility and semi−invertibility of square matrices over commutative semirings, including the links between the basis of semilinear spaces and semi−invertibility. A sufficient condition for semi−invertibility of square matrices over commutative semirings is obtained.

Keywords: commutative semirings, invertible matrix, semi−invertible matrix, semilinear space.

1. Introduction

A semiring is an algebraic structure $(S,+,\cdot)$ in which $(S,+,0)$ is a commutative monoid, $(S,\cdot,1)$ is a monoid, $1 \neq 0$, $\cdot$ is distributive over +, and $for\ all\ r \in S: r \cdot 0 = 0 \cdot r = 0$. A semiring $(S,+,\cdot)$ is called comm− utative if it is multiplicatively commutative. Note that $(S,+,\cdot,0,1)$ is called a semiring with zero 0 and identity 1.

In this paper, we try to relate the semi−invertible matrix with invertible matrix over commutative semirings, and give the sufficient conditions for semi−invertibility of square matrices over commutative semirings. Also, we try to relate the semi−invertible matrix with the basis of semilinear space of $n-$

dimensional vectors over commutative semirings. The paper is organized as follows.

Some definitions, some propositions, notions and some theorems are given in section 2. In the third section we introduce some theorems which give us the sufficient conditions for square matrices over commutative semirings to be semi–invertible. The section ends with an application in solving a system of simultaneous linear equations over commutative semiring which can not be embedded in a ring. Note that, a semiring S can be embedded in a ring iff S is additively cancellative (i.e., $a+c=b+c$ implies $=b\ for\ any\ a,b,c\ \in S$).

2. Preliminaries

Let S be a semiring and $M_n(S)$ be the semiring of $n\times n$ square matrices over S.

***Definition* 1.** Let $A=\left(a_{ij}\right)\in M_n(S)$. Then, the positive determinant $|A|^+$ and the negative determinant $|A|^-$ of A are defined as follows.

$$|A|^+=\sum_{i=1}^{n}a_{ij}\left|A_{ij}\right|^{(+)}\quad \text{and}\quad |A|^-=\sum_{i=1}^{n}a_{ij}\left|A_{ij}\right|^{(-)},$$

where A_{ij} is the matrix in $M_{n-1}(S)$ obtained by deleting the *i*th row and the *j*th column of A. $\left|A_{ij}\right|^+$ and $\left|A_{ij}\right|^-$ are called the positive minor and negative minor, respectively. $\left|A_{ij}\right|^{(+)}(\left|A_{ij}\right|^{(-)})$ means $\left|A_{ij}\right|^{+}(\left|A_{ij}\right|^{-})$ when $i+j$ is even, and it means $\left|A_{ij}\right|^{-}(\left|A_{ij}\right|^{+})$ when $i+j$ is odd.

***Lemma* 1.[4]** Let $A\in M_n(S)$, then $|A^T|^+=|A|^+,|A^T|^-=|A|^-$.

***Lemma* 2.[4]** Let $A=\left(a_{ij}\right)\in M_n(S)$, then for $r\neq t$,

$$\sum_{k=1}^{n}a_{rk}|A_{tk}|^{(+)}=\sum_{k=1}^{n}a_{rk}|A_{tk}|^{(-)}.$$

***Theorem* 1. [1]** Let S be a commutative semiring and let $A,B\in M_n(S)$. If $AB=I_n$ then $BA=I_n$.

***Definition* 2.** A matrix $A\in M_n(S)$ is said to be invertible if $AB=I_n$=BA for some $B\in M_n(S)$.

Note that, for any tow matrices $A,B\in M_n(S)$, we have $|AB|^+=|A|^+|B|^++|A|^-|B|^-+r$ and $|AB|^-=|A|^+|B|^-+|A|^-|B|^++r$ for some $r\in S$. Also, $|I_n|^+=1$ and $|I_n|^-=0$.

***Theorem* 2. [3]** If $A, B \in M_n(S)$, then $|AB|^+ + |A|^+|B|^- + |A|^-|B|^+ = |AB|^- + |A|^+|B|^+ + |A|^-|B|^-$.

***Proposition* 1. [4]** Let $A \in M_n(S)$ be an invertible matrix. Then $|A|^+ \neq |A|^-$.

***Definition* 3.** Let $\mathcal{A} = (A, +_A, 0_A)$ be a commutative monoid and $\mathcal{L} = (S, +, \cdot, 0, 1)$ is a commutative semiring. If an external multiplication $\bullet : S \times A \to A$ such that :

(i) $for\ all\ r, s \in S, a \in A : (r \cdot s) \bullet a = r \bullet (s \bullet a)$;
(ii) $for\ all\ r \in S, a, b \in A: r \bullet (a +_A b) = r \bullet a +_A r \bullet b$;
(iii) $for\ all\ r, s \in S, a \in A : (r + s) \bullet a = r \bullet a +_A s \bullet a$;
(iv) $for\ all\ a \in A: 1 \bullet a = a$;
(v) $for\ all\ r \in S, a \in A: 0 \bullet a = r \bullet 0_A = 0_A$,

is defined, then $\mathcal{A}$ is called a left $\mathcal{L}$−semimodule.

The definition of a right $\mathcal{L}$−semimodule is analogous, where the external multiplication is defined as a function $A \times S \to A$. An $\mathcal{L}$−semimodule is both right as well as left $\mathcal{L}$−semimodule.

***Definition* 4.** Let $\mathcal{L} = (S, +, \cdot, 0, 1)$ be a commutative semiring. Then a semimodule over $\mathcal{L}$ is called an $\mathcal{L}$−semilinear space. The elements of a semilinear space will be called vectors and elements of S scalars.

***Example* 1.** Let $\mathcal{L} = (S, +, \cdot, 0, 1)$ be a commutative semiring. For each $n \geq 1$, let $V_n(S) = \{(r_1, \cdots, r_n)^T : r_i \in S, 1 \leq i \leq n\}$. Then, $V_n(S)$ becomes a $\mathcal{L}$ −semilinear space if the operations are defined as follows:

$$(r_1, \cdots, r_n)^T + (s_1, \cdots, s_n)^T = (r_1 + s_1, \cdots, r_n + s_n)^T;$$

$r \bullet (r_1, \cdots, r_n)^T = (r \cdot r_1, \cdots, r \cdot r_n)^T$,

For all $r, r_i, s_j \in S$. We denote this semilinear space by $\mathcal{V}_n$ and we call it the semilinear space of n−dimensional over $\mathcal{L}$.

***Definition* 5.** Let $\mathcal{A}$ be an $\mathcal{L}$−semilinear space. For $\lambda_1, \cdots, \lambda_n \in S, r_1, \cdots, r_n \in A$, the element $\lambda_1 r_1 +_A \cdots +_A \lambda_n r_n \in A$ is called a linear combination of vectors $r_1, \cdots, r_n \in A$.

***Definition* 6.** Let $\mathcal{A}$ be an $\mathcal{L}$−semilinear space. Vectors $r_1, \cdots, r_n \in A$, where $n \geq 2$ are called linearly independent if none of these vectors can be expressed as a linear combination of others. Otherwise, we say that vectors

$r_1, \cdots, r_n$ are linearly dependent. A single non−zero vector is linearly independent.

Definition 7. A nonempty subset G of vectors from A is called a generating set if every vector from A is a linear combination of vectors from G.

Definition 8. A linearly independent generating set is called a basis.

Definition 9. A matrix $A \in M_n(S)$ is said to be semi−invertible if there exist $A_1, A_2 \in M_n(S)$ such that $I_n + AA_1 = AA_2$ and $I_n + A_1A = A_2A$.

Proposition **2.[6]** The set $\{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathcal{V}_n$ where

$$e_1 = (1, 0, \cdots, 0)^T,$$
$$e_2 = (0, 1, 0, \cdots, 0)^T,$$
$$\vdots$$
$$e_n = (0, 0, \cdots, 1)^T.$$

3. Main Results

In this section we prove the main results.

Theorem 3. Let S be a commutative semiring, and $A \in M_n(S)$. If A is invertible then A is semi−invertible.

Proof. Suppose that A is invertible. That means, there exists a matrix $B \in M_n(S)$ such that $AB = BA = I_n$. In this case $|A|^+ \neq |A|^-$, because if $|A|^+ = |A|^-$ then $|AB|^+ = |AB|^-$, but $|AB|^+ = |I_n|^+ = 1$ and $|AB|^- = |I_n|^- = 0$ which is a contradiction. Thus $|A|^+ \neq |A|^-$. Now, by theorem 2 we have $|AB|^+ + |A|^+|B|^- + |A|^-|B|^+ = |AB|^- + |A|^+|B|^+ + |A|^-|B|^-$, so we get, $1 + |A|^+|B|^- + |A|^-|B|^+ = |A|^+|B|^+ + |A|^-|B|^-$. Let $r = |B|^-$ and $s = |B|^+$, we get $1 + r|A|^+ + s|A|^- = s|A|^+ + r|A|^-$.

Let $A_1 = rA^+ + sA^-$ and $A_2 = sA^+ + rA^-$. Let $D = A^T, A^+ = \left|D_{ij}\right|^{(+)}$, $A^- = \left|D_{ij}\right|^{(-)}$. If $A_1 = (c_{ij})$, then $A_1A = \left(f_{ij}\right)$, where

$$f_{ij} = \sum_{k=1}^{n} c_{ik}a_{kj} = \sum_{k=1}^{n} a_{kj}c_{ik}$$
$$= \sum_{k=1}^{n} a_{kj}\left(r|D_{ik}|^{(+)} + s|D_{ik}|^{(-)}\right)$$
$$= r\sum_{k=1}^{n} d_{jk}\,|D_{ik}|^{(+)} + s\sum_{k=1}^{n} d_{jk}\,|D_{ik}|^{(-)}$$

Where $D = (d_{ij}), A = (a_{ij})$. Then by definition of $|A|^+$ and $|A|^-$, lemma 1, lemma 2, we get

$$A_1A = f_{ij} = \begin{cases} (r+s)\sum_{k=1}^{n} d_{jk}|D_{ik}|^{(+)}, & i \neq j, \\ r|D|^+ + s|D|^- = r|A|^+ + s|A|^-, & i = j. \end{cases}$$

Similarly, $A_2A = (g_{ij})$, where

$$A_2A = g_{ij} = \begin{cases} (r+s)\sum_{k=1}^{n} d_{jk}|D_{ik}|^{(+)}, & i \neq j, \\ s|A|^+ + r|A|^-, & i = j. \end{cases}$$

Thus $1 + f_{ii} = g_{ii}$ for all $i = 1,2,\cdots,n$ and $f_{ij} = g_{ij}$ for all $i \neq j$. Therefore, $I_n + A_1A = A_2A$. Similarly, we can show that $I_n + AA_1 = AA_2$. Hence, A is semi–invertible. ■

***Corollary* 1.** Let $A \in M_n(S)$. If $B = I_n$, then $1 + |A|^+|B|^- + |A|^-|B|^+ = |A|^+|B|^+ + |A|^-|B|^-$.

***Corollary* 2.** Let $A \in M_n(S)$, and S be a commutative semiring satisfying condition: for each $a \neq b, a, b \in S$, there exist $r, s \in S$, such that $1 + ra + sb = sa + rb$. If $|A|^+ \neq |A|^-$,then A is semi–invertible.

***Theorem* 4.** Let $A \in M_n(S), \mathcal{V}_n = (V_n(S), +, 0_{n\times1})$, where $0_{n\times1} = (0,\cdots,0)^T$.If the set of the column vectors of A is a basis of $\mathcal{V}_n$, then A is semi–invertible.

***Proof* .** Suppose that $r_1, r_2, \cdots, r_n$ are the column vectors of A, then the set $\{r_1, r_2, \cdots, r_m\}$ is a basis of $\mathcal{V}_n$, and every unit vector $e_i \in V_n(S)$ can be represented by

$$e_i = \lambda_{1i}r_1 + \cdots + \lambda_{ni}r_n = (r_1, r_2, \cdots, r_n)\begin{pmatrix} \lambda_{1i} \\ \vdots \\ \lambda_{ni} \end{pmatrix},$$

where the coefficients $\lambda_{1i}, \cdots, \lambda_{ni} \in S, i = 1, \cdots, n$, i.e.,

$$I_n = (e_1, \cdots, e_n) = (r_1, \cdots, r_n)\begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \vdots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} = AB,$$

where $B = (\lambda_{ij})_{n\times n}$. Then by theorem 1 and definition 2, A is invertible, hence by theorem 3, A is semi–invertible.

■

***Corollary* 3.** If all unit vectors of $\mathcal{V}_n$ $(\{e_1, \cdots, e_n\})$ can be represented by a linear combinations of the column vectors of A, then A is semi–invertible.

Example 2.

Let $S = (\mathbb{Q}_+, max, \cdot)$ be a commutative semiring of nonnegative rationales satisfying the condition: for each pair $a \neq b, a, b \in S$, there exist $r, s \in S$, such that $1 + ra + sb = sa + rb$. Solve the system

$$x + 2y + z = 9$$
$$7x + 4y + 3z = 35$$
$$3x + 5y + 2z = 20,$$

where $x, y, z \in S$.

***Solution*.** We can write the last system in the form $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 3 & 5 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 9 \\ 35 \\ 20 \end{pmatrix}.$$

$|A|^+ = 35 \neq 28 = |A|^-$, by *corollary* 2, A is semi–invertible, hence $I_n + A_1 A = A_2 A$, where $A_1 = rA^+ + sA^-$, $A_2 = sA^+ + rA^-$, $r, s \in S$ such that

$$1 + 35r + 28s = 28r + 35s. \quad (1)$$

We have $+A_1 AX = A_2 AX$, which implies

$$X + A_1 B = A_2 B. \quad (2)$$

Note that,

$$A^+ = \begin{pmatrix} 8 & 5 & 6 \\ 9 & 2 & 7 \\ 35 & 6 & 4 \end{pmatrix}, A^- = \begin{pmatrix} 15 & 4 & 4 \\ 14 & 3 & 3 \\ 12 & 5 & 14 \end{pmatrix}, A^+B = \begin{pmatrix} 165 \\ 140 \\ 315 \end{pmatrix}, A^-B = \begin{pmatrix} 140 \\ 126 \\ 280 \end{pmatrix}.$$

Now, (1) is satisfied by $r = 1/315$, $s = 1/35$. We put these values in (2), we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 4.0 \\ 3.6 \\ 8.0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 9 \end{pmatrix},$$

which implies $x = 5, y = 4, z = 9$.

4. Conclusion and Fuser Work

In this paper we study a semi–invertability of matrices over commutative semirings which is a very interesting topic in an abstract algebra, that was: by relating the semi−invertible matrix with invertible matrix over commutative semirings, also we gave the sufficient conditions for semi−invertibility of square matrices over commutative semiring, and finally we related the semi−invertible matrix with the basis of semilinear space of n −dimensional vectors over commutative semi−rings.

The open question here is that: How we can solve the system of a liner algebraic equations over commutative semirings by using a semi−invertible matrices?.

References :

[1] Ghosh S., *Matrices over semirings*, North Holland, Information Sciences 90, 221-230(1996).

[2] Paul H., *Inversive Semirings*, J. Austral. Math. Soc. 18, 277-288(1974).

[3] Phillip L., Robert E., *Determinant identities over commutative semirings*, Linear Algebra App. 387, 99-132 (2004).

[4] Reutenauer C., Straubing H., *Inversion of Matrices over a commutative Semiring*, J. Algebra 88, 350-360(1984).

[5] Sombatboriboon S., Mora W., Kemprasi Y., *Some results concerning invertible matrices over semirings*, Research Article, Science Asia 37, 130-135 (2011).

[6] Zhao S., Wang X., *Invertible matrices and semilinear spaces over commutative semirings*, Information Sciences 180, 5115-5124 (2010).