Zawia University

Administration of Postgraduate Studies and Training

Faculty of science

Department of Mathematics

Global solutions for the two dimensional Quasi Geostrophic Equation in the Besov spaces $B^{s}_{2,1}(\mathbb{R}^2)$, with $s > \frac{3}{2}$

Submitted by:

Manal Hasan

Supervised by:

Dr. Samira Alamin SULAIMAN

Academic degree: associate professor

A Dissertation Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

2022-2023

جامعة الزاوية
إدارة الدراسات العليا والتدريب
كلية العلوم
قسم الرياضيات

الحل العام لمعادلة شبه المغناطيسية في بعدين في فضاء بزوف الدالي

إعداد الطالبة

منال حسن حسين الغضبان

إشراف الدكتورة

سميرة الامين محمد سليمان

الدرجة العلمية: أستاذ مشارك

قدمت هذه الرسالة لاستكمال متطلبات الإجازة العالية الماجستير

بسم الله الرحمن الرحيم

﴿ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ
بِمَا تَعْمَلُونَ خَبِيرٌ ﴾

صدق الله العظيم

سورة المجادلة / الآية "11"

Acknowledgments

Firstly, I give thanks to ALLAH for the protection and the ability to do work. In my journey towards this degree. I have found a teacher, a friend, an inspiration, a role model, and a pillar of support in my Guide, **Dr. Samira Alamin** for all her best and valuable advice, time and sources to help me achieve the best results.

I also thank my family who encouraged me and prayed for me throughout the time of my research. This thesis is heartily dedicated to my mother, who took the lead to heaven before the completion of this work.

Finally, I thanks to everyone who helped me to complete this work.

The researcher

Dedication

I dedicate my dissertation to my father and my mother who always believed in me more than, I have ever believed in myself.

To all member of my family and my friends.

Abstract

The main goal of this thesis is to study the two-dimensional quasi-geostrophic equation. This equation serves as two-dimensional models arising in geophysical fluid dynamics.

We aim to study the global and local existence and uniqueness result for quasi-geostrophic equation with initial data ,that is we are interested to study the following system.

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta + |D|^{\frac{1}{2}} \theta = 0, & (x,t) \in \mathbb{R}^2 \times [0, \infty[\\ div\, v = 0, & \\ \theta|_{t=0} = \theta_0. & \end{cases}$$

The problem is solved in many functional spaces, with small initial data. We will study the paper [15] and apply these results to our case. More precisely, we will prove the problem for $\theta_0 \in B^s_{2,1}$, with $s > \frac{3}{2}$, where $B^s_{2,1}$ is the Besov space given in Chapter II. Finally, combining it with the results of [14] and [15].

Table of contents

General Introduction

1.1 Background and motivation

The two dimensional quasi-geostrophic equation are the form

$$\begin{cases} \partial_t\theta + v.\nabla\theta + |D|^\alpha\theta = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty[\\ div\, v = 0, \end{cases} \qquad (1)$$

where θ is the scalar function represents the potential temperature and the parameter $\alpha \in [0,1]$. The fractional differential operator $|D| = (-\Delta)^{\frac{1}{2}}$ is defined by its Fourier transform

$$\mathcal{F}(|D|v) = |\xi|\mathcal{F}(v),$$

The $2D$ velocity field $v = (v^1,\ v^2)$ is determined by Riesz transform $R_i, \forall i = 1,2$ of θ, that is

$$v = \left(-\frac{\partial_2}{|D|}\theta\, , \frac{\partial_1}{|D|}\theta\right) := (-R_2\theta\, ,\ R_1\theta).$$

The differential operator $v \cdot \nabla$ is defined respectively by

$$v \cdot \nabla = \sum_{i=1}^{2} v^i \cdot \partial_i.$$

and the operator $div\, v$ is defined by

$$div\, v = \sum_{i=1}^{2} \partial_i v^i$$

The first equation of (1) serves as a $2D$ models arising in geophysical fluid dynamic [20] and the second equation $div\, v = 0$, describe the incompressibity of the fluid.
The interesting is to study the global existence results for the initial value problem (IVP) for equation (1) with

$$\theta|_{t=0} = \theta_0(x), \qquad (2)$$

is specified, that concerned with global existence results for solutions of the $(QG)_\alpha$:

$$\begin{cases} \partial_t\theta + v \cdot \nabla\theta + |D|^\alpha\theta = 0 \\ div\, v = 0 \\ \theta|_{t=0} = \theta_0. \end{cases} \qquad (QG)_\alpha$$

In addition, note that the problem is only solved with a smallness initial data.

There exist three cases for α in this problem.

(1) Sub critical case $\alpha > 1$

(2) Critical case $\alpha = 1$

(3) Super critical case $\alpha < 1$

In this category, we precise the notation of critical spaces: let θ be a solution of $(QG)_\alpha$ and $\beta > 0$, then $\theta_\beta(t,x) = \frac{1}{\beta^{1-\alpha}}\theta(\beta^\alpha t, \beta x)$ is also solution of $(QG)_\alpha$.

Before going to these cases, let us firstly give the definition of the Besov spaces and Sobolev spaces, see [2] and [6].

Definition of Besov spaces

We say that a function f in the Besov spaces $B^s_{p,r}$, if $\|f\|_{B^s_{p,r}} < \infty$, where

$$\|f\|_{B^s_{p,r}} := \left(\sum_q 2^{qsr}\left\|\Delta_q f\right\|^r_{L^p}\right)^{\frac{1}{r}}.$$

The bloc dyadic operator Δ_q, see chapter II.

Note that we can define also the Sobolev and Holder spaces by

$$B^s_{2,2} = H^s, \quad B^s_{\infty,\infty} = C^s.$$

We turn now to the cases of α:

Case 1 : Sub critical case $(\alpha > 1)$, the problem of global existence and uniqueness for arbitrary initial data is established in various function spaces we refer to [10]

Case 2 : Critical case $(\alpha = 1)$, the authors in [9] showed the global existence in Sobolev space H^1 under smallness assumption on $\|\theta_0\|_{L^\infty}$, but the uniqueness is proved for initial data in H^2. Many other relevant results can be found in [1], [17], [18].

Case 3 : Super-critical case $(\alpha < 1)$, we had only global results for small initial data. In [5], the global existence and uniqueness are established for data in critical Besov space $B^{2-\alpha}_{2,1}$ with a small norm of initial data. This result was improved by [16] for small initial data in $H^s, s \geq 2-\alpha$. Wu [25] proved the global existence and uniqueness for small initial data in $C^r \cap L^q$, with $r > 1$ and $q \in]1, +\infty[$, where C^r is Holder space.

Also the authors in [26] was established the global well posedness result for small initial data in $B^{s}_{2,\infty} \cap B^{s}_{p,\infty}$, with $s > 2-\alpha$ and $p = 2^N$.

Finally, we mentioned the paper [26], where the authors proved the problem of existence and uniqueness in this case, that is in the super-critical case $(\alpha < 1)$ with initial data in inhomogeneous critical Besov space $B^{1+\frac{2}{p}-\alpha}_{p,1}$, with $p \in [1,\infty]$.

1.2 Main aims of the thesis

There are numerous study for theses three cases and in a different functional spaces. In this research, we interested to the study of the last case of α, that is for the super-critical case $(\alpha < 1)$. Specially, we will prove a smoothing effects on Besov space $\dot{B}^{s+\frac{1}{2r}}_{2,1}, r \in [1,+\infty]$. After this, we prove the global existence and uniqueness in $B^{s}_{2,1}(\mathbb{R}^2)$, with $s > \frac{3}{2}$, and we note that the proof of our results are different from [14], [15] and others references. Our additions are in the proof of our results. Finally, we combine their results with the results of [14] and [15].

In this step, we need to recall here the **Beal-Kato and Majda** criterion which is

the main argument to get global well–posedness results with smooth initial data, see[3].

Therefore, to obtain the global existence, it suffice to use the BKM criterion, which allows us to obtain the L^∞ norm of the vorticity ω, and then to obtain the Lipchitz norm of the velocity $\|\nabla v\|_{L^\infty}$.

This BKM criterion ensuring that the development of finite-time singularity is related to the blow up of the L^∞ norm of the verticity $(\omega = curl v)$, that is

$$T < +\infty \qquad \text{Iff} \qquad \int_0^T \|\omega(t)\|_{l^\infty} dt = \infty,$$

where the vorticity ω in dimension two define as the scalar function

$$\omega = \nabla . v = \partial_1 v^2 - \partial_2 v^1 .$$

Our result reads as follows.

Theorem

Let $\theta_0 \in B^{s}_{2,1}, s > \frac{3}{2}$, then there exists $T > 0$ such that the $(QG)_{\frac{1}{2}}$ equation has a unique solution θ such that

$$\theta \in C([0,T]; B^{s}_{2,1}) \cap L^1_T \dot{B}^{s+\frac{1}{2}}_{2,1} .$$

In other words, there are exists $\beta > 0$, such that $\|\theta_0\|_{\dot{B}^{1}_{\infty,1}} \leq \beta$, therefore we have $T = \infty$.

We note that in the proof our result, we use some embedding's between Besov space and some functional spaces combined with smoothing effects.

1.3 Organisation of the thesis

The objectives of the thesis can be summarized as follows:

In chapter I, we introduce some notations, give the review of functions and mathematical concepts. Besides, we recall some functional spaces and finally we present some well-known results.

In chapter II, we recall some basic results on Littlewood-Paley theory and give the definition of some functional spaces as Besov space, Holder and Sobolev spaces. Finally, we give some useful lemma as Bernstein inequality for a tempered distribution $u \in \mathcal{S}$ (where $\mathcal{S}$ is Schwartz space defined in chapter I).

In chapter III, we give some useful estimates for any smooth solution of linear transport-diffusion model given by

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta + |D|^{\alpha} \theta = f \\ \quad div\, v = 0 \\ \quad \theta|_{t=0} = \theta_0 \end{cases} \qquad (TD)_{\alpha}$$

we will discuss two kinds of estimates that will be used in the next chapter. The first is the L^p energy estimate, $\forall p \in [1, \infty]$. Second, I will prove a smoothing effects which is the main result in this chapter.

In chapter IV, we study some results for the super-critical case that is $(\alpha < 1)$. We prove my main result, and we devised the proof into four steps: a priori estimates, global existence, local existence and uniqueness of the solution. Finally, we combine the results of [8], [14] and [15].

Chapter I:
Basic concepts

1.1 Introduction

We will present some notations that will be used later. In addition, we review some definitions and mathematical concepts for the functions. we illustrate some subjects related to our work of the thesis and give some well-known results.

1.2 Notations

In this section, we introduce some notations:

1- For any positive A and B, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.

2-For any A, B and C, we define the commutator $[A,B]C$ by the

$$[A,B]C = A(BC) - B(AC).$$

3- For any two spaces X and Y, any function $f \in Y$, we say that $X \hookrightarrow Y$, if there exists a positive constant $C > 0$, such that

$$\|f\|_Y \leq C\|f\|_X \, .$$

4- For the usual Lebegue space $L^p, p \in [1,\infty]$, which defined in Definition 1.4.4 below, we will use the notation

$$\|f\|_{L_T^p L^Z} := \left(\int_0^T \|f(\tau)\|_Z^p d\tau \right)^{\frac{1}{p}}, \quad \forall\, T > 0.$$

5- We will introduce the following notation: we denote by

$$\Phi_l(t) = \underbrace{C_0 exp\big(\ldots exp(C_0\, t)\big)}_{n-times}$$

where C_0, depends only on the initial data and its value may from line to line up to some absolute constants. We will make an intensive use of the following trivial facts

$$\int_0^t \Phi_l(\tau)d\tau \leq \Phi_l(t) \quad \text{and} \quad \exp\left(\int_0^t \Phi_l(\tau)d\tau\right) \leq \Phi_{l+1}(t).$$

1.3 Review of functions and mathematical concepts:

In this section, we give some definitions for functions and mathematical concepts.

Definition 1.3.1

We say that f is bounded if there exists a positive number $M > 0$ such that for any $x \in M$,

$$|f(x)| \leq M.$$

Definition 1.3.2

Let f be a real valued function, we say that f satisfies the Lipschitz condition if, there exists a constant C such that for every x and y, we have

$$|f(x) - f(y)| \leq C|x - y|.$$

Definition 1.3.3

Let $f : X \to X$ be a real valued function, we denote $\|f\|$ is the norm of f and is defined for every $x \in X$, by

$$\|f\| = sup_{\|x\|=1}|f(x)|.$$

Definition 1.3.4

The function f is called continuous at $x_0 \in X$, if for any $\varepsilon > 0$, there exists $\delta > 0$ depend on x, and x_0, such that

$$\|f(x) - f(x_0)\| < \varepsilon \text{ Whenever } \|x - x_0\| < \delta.$$

Definition 1.3. 5

Let (M, A) be a smooth manifold, and $\mathrm{f}: \mathrm{M} \to \mathbb{R}$, a function.

(1) We say f is smooth at $\mathrm{p} \in \mathrm{M}$ if there exists a chart $\{ \varphi_\alpha, \mathrm{U}_\alpha, \mathrm{V}_\alpha\} \in \mathrm{A}$ with $\mathrm{p} \in \mathrm{U}_\alpha$, such that the function $\mathrm{f} \circ \varphi_\alpha^{-1}: \mathrm{V}_\alpha \to \mathbb{R}$ is smooth at $\varphi_\alpha(\mathrm{p})$.

(2) We say f is a smooth function on M if it is smooth at every $\mathrm{x} \in \mathrm{M}$.

Definition 1.3.6

We say that a real valued function f is smooth on the closed interval $[a, b]$, if the function f and its derivative are continuous on $[a, b]$.

We note that f is smooth in $\mathbb{R}$ if and only if f is smooth in all interval on $\mathbb{R}$.

Definition 1.3.7

A continuous map $f: X \to Y$ is homeomorphism, if it is bijective and its inverse is continuous.

Definition 1.3.8

For any integrable function f, we denote by $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of f, where

$$\hat{f}(\xi) = \mathcal{F}(f) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi}\, dx.$$

Moreover, the inverse Fourier transform is given by

$$\hat{f}(\xi) = \mathcal{F}^{-1}(\hat{f}(\xi)) .$$

$$\coloneqq \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi}\, d\xi.$$

Definition 1.3.9

For any two functions f and g, we define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(\zeta) g(x - \zeta) d\zeta.$$

Definition 1.3.10

We define the flow associate to the velocity v by the following:

$$\psi(t, x) = x + \int_0^t v(\tau, \psi(\tau)) d\tau .$$

Definition 1.3.11

For any function f, and any points x and x_1. Then the Taylor formula of the function f is given by

$$f(x) = (x - x_1) \int_0^1 \acute{f}(sx) ds.$$

1.4 Some functional spaces

Here, in this section, we define some functional spaces.

Definition 1.4.1

The space $C(D)$ is the space of all continuous functions on any region D, with norm $\|.\|_\infty$ defined as

$$\|f(x)\|_\infty = max_{x\in D}|f(x)|$$

Definition 1.4.2

The space C_0^∞ is the space of all continuous function f and differentiable such that the space is compact.

Definition 1.4.3 (Schwartz space)

The Schwartz space $S(\mathbb{R}^d)$ is the space of smooth functions f on $\mathbb{R}^d$ such that $f \in C^\infty$, and for all α and for any $N \in \mathbb{N}$, there exists a constant $C_{N,\alpha}$ depend on N and α, such that

$$|\partial^\alpha f(x)| \leq \frac{C_{N,\alpha}}{(1+|x|)^{-N}}.$$

Remark 1.4.1 we have the relation between the spaces $C_0^\infty$and Schawrts space S which given by the following embedding:

$$C_0^\infty \hookrightarrow S.$$

Definition 1.4.4 (Lebesgue space L^p)

We define the usual Lebesgue space $L^p(\mathbb{R}^d)$, with $p \in [1,\infty[$, by the space of all continuous real valued functions f on $\mathbb{R}^d$, with norm defined as

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}|f(x)|^p dx\right)^{\frac{1}{p}} < \infty,$$

and for $p = \infty$, we have

$$\|f\|_{L^\infty} = sup_x|f(x)|.$$

Definition 1.4.5 (space l^p)

For any function f, we define the $l^p(\mathbb{R}^d)$,with $p \in [1,\infty)$, norm of f by

$$\|f\|_{l^p(\mathbb{R}^d)} = (\sum \ |f(x)|^p)^{\frac{1}{p}} \text{ , for any } x \in \mathbb{R}^d$$

Definition 1.4.6

Let $s \in \mathbb{R}$, then the inhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions u such that $\hat{u} \in L^2_{\text{loc}}(\mathbb{R})^d$, and

$$\|u\|^2_{H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty .$$

Definition 1.4.7

Let $s \in \mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the space of tempered distributions u over $\mathbb{R}^d$, such that the Fourier transform of which belongs to $L^1_{Loc}(\mathbb{R}^d)$ and satisfies

$$\|u\|^2_{\dot{H}^s} = \int_{\mathbb{R}^d} |\xi|^{2s} \, |\hat{u}(\xi)|^{2s} \, d\xi \ < \infty.$$

1.5 Some Well-known results:

In this section, we give some well-known results as young inequality and young inequality for convolution. Also recall Holder triangle, and Cauchy Schwartz inequalities. Finally, we give lemma of Gronwall, and Leibnitz formula for derivatives. Also, we recall the Pareseval identity.

Lemma 1.5.1 (triangle inequality)

For any two functions f and g, we have

$$\|f(x)+g(x)\| \leq \|f(x)\| + \|g(x)\|.$$

Lemma 1.5.2 (Young inequality)

For every $a, b > 0$ and $r, s > 0$ then we have the following inequality

$$a\,b \leq \frac{a^r}{r} + \frac{b^s}{s}.$$

Lemma 1.5.3 (Young inequality for convolution)

For any two functions f and g, such that $f \in L^c$ and $g \in L^a$ and for any constants $(a,b,c) \in [1,\infty]^3$, such that

$$1 + \frac{1}{b} = \frac{1}{c} + \frac{1}{a} .$$

Then $f * g \in L^b$, and there exists a positive constant C, such that

$$\|f * g\|_{L^b} \leq C\|f\|_{L^c}\|g\|_{L^a}.$$

Lemma 1.5.4 (Holder inequality)

If (f, g) belongs to $L^P \times L^q$ for any $(p, q, r) \in [1, \infty]^3$ and such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then fg belongs to L^r and satisfies

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Lemma 1.5.5 (Cauchy Schwartz inequality)

Let f and g be two real continuous functions on the closed interval $[a, b]$. Then the Cauchy Schwartz inequality is given by

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

This gives that,

$$\|fg\|_{L^1} \leq \|f\|_{L^2}\ \|g\|_{L^2}.$$

Lemma 1.5.6 (Gronwall's inequality)

Let f is a nonnegative continuous function on $[0, t]$, a is a real number and let A be a continuous function on $[0, t]$. Suppose also that:

$$f(x) \leq a + \int_0^t A(\tau)f(\tau)d\tau.$$

Then we have

$$f(t) \leq a\ exp\left(\int_0^t A(\tau)d\tau\right).$$

Lemma 1.5.7 (Leibnitz formula)

Let $\alpha\ (x),\ \beta(x)$ and $f(x, t)$ any three functions, then we have Leibnitz's formula:

$$\frac{d}{dx}\int_{\alpha\ (x)}^{\beta(x)} f(x, t)\ dt = f\big(x,\ \beta(x)\big)\ \acute{\beta}(x) f\big(x,\ \alpha(x)\big)\ \acute{\alpha}(x)$$

$$+ \int_{\alpha\,(x)}^{\beta(x)} \frac{\partial}{\partial x} f(x,t)dt.$$

Lemma 1.5.8 (Parseval Identity)

For any two functions $f(x)$ and $g(x)$, we have the Parseval identity

$$\langle f(x), g(x) \rangle = \langle f(x), \overline{g(x)} \rangle.$$

Chapter II:
Littlewood Paley operators

2.1 Introduction

Littlewood–Paley theory is a localization procedure in frequency space. The Interesting feature of this localization is that the derivatives (or, more generally, Fourier multipliers) act almost as homotheties on distributions whose Fourier transforms are supported in a ball or an annulus. In this chapter, we define the dyadic decomposition of the space $\mathbb{R}^2$ and recall the Littlewood-Paley operators. We will prove a Bernstein inequality for a tempered distribution u with a bloc dyadic $\dot{\Delta}_q$ and S_q (see definition below). we also discuss the definition of some functional spaces, and in the next section introduce the (homogeneous) paradifferential calculus, and some results which need later. The definition of homogeneous and inhomogeneous Besov spaces are detailed, see $[2],[6],[7]$ and $[24]$. Finally, we give the way that the product acts on Besov spaces.

2.2 Dyadic decomposition

To introduce Besov spaces which are generalization of Sobolev spaces, we need to recall the dyadic decomposition of the whole space see Chemin $[2]$ and $[6]$. .
We review some important lemmas that will be used constantly in the research.

Definition 2.2.1

There exists two nonnegative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\})$ such that,

1- $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \ \forall\, \xi \in \mathbb{R}^2$,

2- $\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \ \forall\, \xi \in \mathbb{R}^2 \setminus \{0\}$,

3- $|p-q| \geq 2 \Rightarrow supp\ \varphi(2^{-p}) \cap supp\ \varphi(2^{-q}) = \emptyset,$

4- $q \geq 1 \Rightarrow supp\ \chi \cap\ supp\ \varphi(2^{-q}.) = \emptyset.$

Definition 2.2.2

Let $v \in S'(\mathbb{R}^2)$, we define the nonhomogeneous Littlewood-Paley operators by,

$$\Delta_{-1} v = \chi(D) v, \ ,$$

$$\Delta_q v = \varphi(2^{-q}D)v, \qquad \forall q \geq 0$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $h_1 = \mathcal{F}^{-1}(\chi)$. Then we can write the operator $\Delta_q v$ as

$$\Delta_q v = 2^{2q} \int_{\mathbb{R}^2} h(2^q \xi) v(\xi) d\xi .$$

and

$$S_q v = \sum_{-1 \leq p \leq q-1} \Delta_p v.$$

$$= 2^{2q} \int_{\mathbb{R}^2} h_1(2^q \xi) v(\xi) d\xi ,$$

and

$$\Delta_{-1} v = S_0 v, \qquad \Delta_q v = 0, \qquad \forall q \leq -2.$$

Definition 2.2.3

We define the homogeneous operators by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q v = \varphi(2^{-q}D)v,$$

and

$$\dot{S}_q v = \sum_{p \leq q-1} \dot{\Delta}_p v.$$

Remarks 2.2.1

1- We decompose v as :

$$v = \Delta_{-1} v + \textstyle\sum_{q \geq 0} \Delta_q v$$

$$= \sum_{q \geq -1} \Delta_q v, \quad \forall\, v \in S'(\mathbb{R}^2) .$$

2- We also write

$$v = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q v , \qquad \forall\, v \in S'(\mathbb{R}^2)/\mathrm{P}(\mathbb{R}^2),$$

where $\mathrm{P}(\mathbb{R}^2)$ is the space of polynomials.

3- The Littlewood-Paley decomposition satisfies the property of almost orthogonally:

For any $u, v \in S'(\mathbb{R}^2)$,

$$\Delta_p \Delta_q u = 0 \quad \text{If} \quad |p-q| \geq 2$$

$$\Delta_p\left(S_{q-1} u \, \Delta_q v\right) = 0 \text{ If } |p-q| \geq 5.$$

4- The operators Δ_q and S_q map continuously L^p into itself uniformly with respect to q and p.

5- We have $\Delta_q = \dot{\Delta}_q$, $\forall\, q \in \mathbb{N}$ and S_q coincides with $\dot{S}_q$ on tempered distributions modulo polynomials.

The following result is needed, see [6] and [7].

Lemma 2.2.1

For every function $f \in S$, where S is the space of Schwartz such that $f \in L^1 \cap L^\infty$ and for every $1 < c < \infty$, then we have $f \in L^c$ and $(1 + |.|^2)^d \partial^\alpha$ is bounded.

A further important result that will be constantly used here so called Bernstein inequalities. Note that [7] proved this inequality for any tempered distribution u, and the supervisor of this thesis S. Sulaiman [8] and [23], proved the same inequality for the bloc dyadic S_q and $\dot{\Delta}_q$, we will give here a complete proof.

Lemma 2.2.2 (Bernstein lemma)

There exists a constant $C > 0$ such that for all $q \in \mathbb{Z}$, $k \in \mathbb{N}$ and for every tempered distribution u we have

$$sup_{|\alpha|=k} \left\|\partial^\alpha S_q u\right\|_{L^b} \leq C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \left\|S_q u\right\|_{L^a}, \qquad b \geq a \geq 1 \quad (2.1)$$

$$C^{-k} 2^{qk} \left\|\dot{\Delta}_q u\right\|_{L^a} \leq sup_{|\alpha|=k} \left\|\partial^\alpha \dot{\Delta}_q u\right\|_{L^a} \leq C^k 2^{qk} \left\|\dot{\Delta}_q u\right\|_{L^a} \qquad (2.2)$$

Proof of $(\mathbf{2.1})$

If $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi \equiv 1$ in the neighbourhood of the ball of center 0 and radius r_1. If also $\varphi_1 \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_1 \equiv 1$ in the neighbourhood of φ, then we have

$$S_q u = \varphi_1(2^{-q} D) S_q u\, .$$

Then, we can write

$$S_q u = \mathcal{F}^{-1}\left(\varphi_1(2^{-q}D)\mathcal{F}\left(S_q u\right)\right) = \mathcal{F}^{-1}(\varphi_1(2^{-q}D)) * S_q u\, .$$

We get by the Fourier transform with a simple calculation,

$$\mathcal{F}^{-1}\big(\varphi_1(2^{-q}D)\big) = \int\limits_{\mathbb{R}^d} \varphi_1(2^{-q}\zeta)e^{ix\zeta}d\zeta = \int\limits_{\mathbb{R}^d} 2^{qd}\varphi_1(\zeta)e^{ix2^q\zeta}d\zeta$$

$$= 2^{qd}\mathcal{F}^{-1}\big(\varphi_1(\zeta)\big) \coloneqq 2^{qd}h(2^q x),$$

Where

$$h(2^q x) = \mathcal{F}^{-1}\big(\varphi_1(\zeta)\big)$$

This gives that

$$S_q u = 2^{qd}h(2^q.) * S_q u \,.$$

Therefore

$$\partial^\alpha S_q u = 2^{q(d+|\alpha|)}\partial^\alpha h(2^q.) * S_q u \qquad (2.3)$$

Taking the L^p norm of (2.3) and applying young inequality for convolution Lemma 1.5.3, we find with $(\frac{1}{b} + 1 = \frac{1}{c} + \frac{1}{a})$,that

$$\left\|\partial^\alpha S_q u\right\|_{L^p} \leq 2^{q(d+|\alpha|)}\|\partial^\alpha h(2^q.)\|_{L^c}\left\|S_q u\right\|_{L^a},$$

$$\leq 2^{q(|\alpha|+d)}2^{-q\frac{d}{c}}\|\partial^\alpha h\|_{L^c}\left\|S_q u\right\|_{L^a}$$

$$\leq 2^{q\left(|\alpha|+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\|\partial^\alpha h\|_{L^c}\left\|S_q u\right\|_{L^a} \quad (2.4)$$

Taking the sup on $|\alpha| = k$, of the inequality (2.4), we obtain

$$sup_{|\alpha|=k}\left\|\partial^\alpha S_q u\right\|_{L^b} \leq \left(2^{q(k+d)\left(\frac{1}{a}-\frac{1}{b}\right)}\right)\|\partial^\alpha h\|_{L^c}\left\|S_q u\right\|_{L^a} \qquad (2.5)$$

It remains now to prove that$\|\partial^\alpha h\|_{L^c} \leq C^k$. For this purpose, we use Lemma 2.2.1, then we have

$$\|\partial^\alpha h\|_{L^c} \leq \|\partial^\alpha h\|_{L^1} + \|\partial^\alpha h\|_{L^\infty} \qquad (2.6)$$

Now since $h \in S$, $h = \mathcal{F}^{-1}\varphi$ and $\varphi \in C_0^\infty(\mathbb{R}^d) \hookrightarrow S$, then I can use Lemma 2.2.1, that h is bounded and $(1 + |.|^2)^d\, \partial^\alpha h$ is also bounded. Therefore, we have

$$\|\partial^\alpha h\|_{L^1} = \int |\partial^\alpha h(x)|dx \leq \int (1 + |.|^2)^{-d}\,(1 + |.|^2)^d|\partial^\alpha h|dx$$

$$\leq \|(1 + |.|^2)^{-d}\|_{L^1}\|(1 + |.|^2)^d\partial^\alpha h\|_{L^\infty}$$

$$\leq C\|(1+|.|^2)^d \partial^\alpha h\|_{L^\infty} \qquad (2.7)$$

Also

$$\|\partial^\alpha h\|_{L^\infty} = sup_x|\partial^\alpha h(x)| \leq sup_x(1+|.|^2)^d|\partial^\alpha h|$$

$$\leq C\|(1+|.|^2)^d \partial^\alpha h\|_{L^\infty} \qquad (2.8)$$

Putting together (2.7) and (2.8) in (2.6), we get

$$\|\partial^\alpha h\|_{L^c} \leq C^2\|(1+|.|^2)^d \partial^\alpha h\|_{L^\infty} \leq C^k, k \in \mathbb{N}\,.$$

This gives in (2.5), that

$$sup_{|\alpha|=k}\left\|\partial^\alpha S_q u\right\|_{L^b} \leq C^k 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\left\|S_q u\right\|_{L^a}.$$

Proof of (2.2) of Lemme 2.2.2

Let $\varphi_1 \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_1 \equiv 1$ in the neighbourhood of φ .Then we have

$$\dot{\Delta}_q u = \varphi_1(2^{-q}D)\dot{\Delta}_q u \qquad (2.9)$$

We take the Fourier transform of (2.9),

$$\mathcal{F}\big(\dot{\Delta}_q u\big) = \varphi_1(2^{-q}\zeta)\mathcal{F}\big(\dot{\Delta}_q u\big)\,,$$

I can Take the inverse Fourier transform, we obtain

$$\dot{\Delta}_q u(x) = \mathcal{F}^{-1}\left(\varphi_1(2^{-q}\zeta)\mathcal{F}\big(\dot{\Delta}_q u\big)(\zeta)\right) \qquad (2.10)$$

where,

$$\varphi_1(2^{-q}\zeta) = \sum_{|\alpha|=k}(i\zeta)^\alpha\,|\zeta|^{-2k}(-i\zeta)^\alpha\varphi_1(2^{-q}\zeta)$$

Putting this last inequality in (2.10), we get

$$\dot{\Delta}_q u(x) = \sum_{|\alpha|=k}\mathcal{F}^{-1}\left((i\zeta)^\alpha|\zeta|^{-2k}(-i\zeta)^\alpha\varphi_1(2^{-q}\zeta)\mathcal{F}\big(\dot{\Delta}_q u\big)(\zeta)\right)$$

$$= \sum_{|\alpha|=k}\mathcal{F}^{-1}\left((i\zeta)^\alpha|\zeta|^{-2k}\varphi_1(2^{-q}\zeta)\mathcal{F}\big(\partial^\alpha\dot{\Delta}_q u\big)(\zeta)\right)$$

$$= \textstyle\sum_{|\alpha|=k}\mathcal{F}^{-1}\Big((i\zeta)^\alpha|\zeta|^{-2k}\varphi_1(2^{-q}\zeta)\Big) * \partial^\alpha\dot{\Delta}_q u\,(x) \qquad (2.11)$$

where,

$$\mathcal{F}^{-1}((i\zeta)^{\alpha}|\zeta|^{-2k}\varphi_1(2^{-q}\zeta) = \int \frac{(i\zeta)^{\alpha}}{|\zeta|^{2k}}\varphi_1(2^{-q}\zeta)e^{ix\zeta}d\zeta$$

$$= \int \frac{(i2^q\zeta)^{\alpha}}{|2^q\zeta|^{2k}}\varphi_1(\zeta)e^{ix2^{qd\zeta}}d\zeta$$

$$= 2^{q(d+|\alpha|-2k}\int \frac{(i\zeta)^{\alpha}}{|\zeta|^{2k}}\varphi_1(\zeta)e^{ix2^{qd}\zeta}d\zeta$$

$$= 2^{q(d+|\alpha|-2k)}\, h_k(2^q x),$$

where

$$h_k(2^q x) = \int \frac{(i\zeta)^{\alpha}}{|\zeta|^{2k}}\varphi_1(\zeta)e^{ix2^{qd}\zeta}d\zeta$$

Then we get in view of (2.11), that

$$\dot{\Delta}_q u(x) = 2^{q(d+|\alpha|-2k)}\, h_k(2^q.) * \partial^{\alpha}\dot{\Delta}_q u$$

This is given by Lemma1.5.3 for convolution, that

$$\left\|\dot{\Delta}_q u\right\|_{L^a} \leq 2^{q(d+|\alpha|-2k)}\|h_k(2^q.)\|_{L^1}\left\|\partial^{\alpha}\dot{\Delta}_q u\right\|_{L^a} \qquad (2.12)$$

Since, we have

$$\|h_k(2^q.)\|_{L^1} = \int |h_k(2^q x)|\, dx$$

Let $y \coloneqq 2^q x$, then we get

$$\|h_k(2^q.)\|_{L^1} = \int |h_k(y)|2^{-qd}dy = 2^{-qd}\|h_k\|_{L^1} \qquad (2.13)$$

Recall that, $h = \mathcal{F}^{-1}\varphi$ and $\varphi \in C_0^{\infty}(\mathbb{R}^d) \hookrightarrow S$, then we have $h \in S$, this gives by using Lemma 2.2.1, h is bounded and $(1+|.|^2)^d\,\partial^{\alpha}h$ is also bounded. Therefore

$$\|h_k\|_{L^1} \leq C^k \qquad (2.14)$$

Putting together (2.13) and (2.14) into (2.12), we find

$$\left\|\dot{\Delta}_q u\right\|_{L^a} \leq C^k 2^{q(d+|\alpha|-2k)}\left\|\partial^{\alpha}\dot{\Delta}_q u\right\|_{L^a} \qquad (2.15)$$

Taking the supremum on $|\alpha| = k$, of the inequality (2.15), yields to

$$C^{-k}2^{qk}\left\|\dot{\Delta}_q u\right\|_{L^a} \leq sup_{|\alpha|=k}\left\|\partial^{\alpha}\dot{\Delta}_q u\right\|_{L^a}.$$

The proof of the lemma is now complete.

The following lemma is useful to our result, see [15], and we will give here the proof.

Lemma 2.2.3

Let f be a function in Schwartz class and ψ a diffeomorphism preserving Lebesgue measure, then for all $p \in [1, +\infty]$, and for all $j, q \in \mathbb{Z}$, we have

$$\left\|\dot{\Delta}_j(\dot{\Delta}_q f o\psi)\right\|_{L^p} \leq C2^{|j-q|}\left\|\nabla\psi^{\alpha(j,q)}\right\|_{L^\infty}\left\|\dot{\Delta}_q f\right\|_{L^p},$$

with

$$\alpha(j,q) = sign(j-q).$$

Proof

To prove this result, we distinguish two cases: $j \geq q$ and $j < q$.

Case 1: $\boldsymbol{j \geq q}$.

For this, we use Bernstein's inequality, to get

$$\left\|\dot{\Delta}_j(\dot{\Delta}_q f o\psi)\right\|_{L^p} \lesssim 2^{-j}\left\|\nabla\dot{\Delta}_j\left(\dot{\Delta}_q f o\psi\right)\right\|_{L^p} \qquad (2.16)$$

It suffices to combine Leibnitz formula again with Bernstein's inequality and Holder inequality.

$$\begin{aligned}\left\|\nabla\dot{\Delta}_j\left(\dot{\Delta}_q f o\psi\right)\right\|_{L^p} &\lesssim \left\|\nabla\dot{\Delta}_q f\right\|_{L^p}\|\nabla\psi\|_{L^\infty} \\ &\lesssim 2^q\left\|\dot{\Delta}_q f\right\|_{L^p}\|\nabla\psi\|_{L^\infty} \qquad (2.17)\end{aligned}$$

Substitute (2.17) into (2.16), get

$$\left\|\dot{\Delta}_j(\dot{\Delta}_q f o\psi)\right\|_{L^p} \lesssim 2^{q-j}\left\|\dot{\Delta}_q f\right\|_{L^p}\|\nabla\psi\|_{L^\infty}.$$

This yields to the desired inequality.

Case 2: $\boldsymbol{j < q}$

Will use the following duality result

$$\left\|\dot{\Delta}_j\left(\dot{\Delta}_q f o\psi\right)\right\|_{L^p} = sup_{\|g\|_{L^{p_1}\leq 1}}\left|\langle\dot{\Delta}_j\left(\dot{\Delta}_q f o\psi\right), g\rangle\right| \qquad (2.18)$$

with $\frac{1}{p} + \frac{1}{p_1} = 1$. Let $\varphi_1 \epsilon C_0^\infty(\mathbb{R}^d)$be supported in a ring and such that $\varphi_1 \equiv 1$ on C. We set $\bar{\dot{\Delta}}_q f \coloneqq \varphi_1(2^{-q}D)f$. Then we have $\dot{\Delta}_q f = \overline{\dot{\Delta}_q}\dot{\Delta}_q f$. Combining this fact with Parseval's identity and the preserving measure by the flow

$$\left|\langle\dot{\Delta}_j(\dot{\Delta}_q f o\psi), g\rangle\right| = \left|\langle\dot{\Delta}_q f, \bar{\dot{\Delta}}_q\ ((\dot{\Delta}_j g)o\psi^{-1})\rangle\right|.$$

Therefore, we obtain

$$\left|\langle\dot{\Delta}_j(\dot{\Delta}_q f o\psi), g\rangle\right| \leq \left\|\dot{\Delta}_q f\right\|_{L^p} \left\|\bar{\dot{\Delta}}_q\ ((\dot{\Delta}_j g)o\psi^{-1})\right\|_{L^{p_1}}$$

This implies in view of Bernstein's inequality and Holder inequality

$$\left|\langle\dot{\Delta}_j(\dot{\Delta}_q f o\psi), g\rangle\right| \lesssim \left\|\dot{\Delta}_q f\right\|_{L^p} 2^{j-q} \|\nabla\psi^{-1}\|_{L^\infty} \left\|\dot{\Delta}_j g\right\|_{L^{p_1}} \qquad (2.19)$$

Substitute (2.19) into (2.18), we get

$$\left\|\dot{\Delta}_j(\dot{\Delta}_q f o\psi)\right\|_{L^p} \lesssim \left\|\dot{\Delta}_q f\right\|_{L^p} 2^{j-q} \|\nabla\psi^{-1}\|_{L^\infty} \|g\|_{L^{p_1}}.$$

It completes the proof.

2.3 Homogeneous and Inhomogeneous Besov space

Now will define the homogeneous and inhomogeneous Besov spaces by using Littlewood-Paley operators. We recall also the definition of Chemin Lerner space and give some results that will need later.

Definition 2.3.1

Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The inhomogeneous Besov space $B^s_{p,r}$ is defined by

$$B^s_{p,r} = \left\{f \in S'(\mathbb{R}^2) \coloneqq \|f\|_{B^s_{p,r}} < \infty\right\},$$

where

$$\|f\|_{B^s_{p,r}} \coloneqq \left\|2^{qs}\left\|\Delta_q f\right\|_{L^p}\right\|_{\ell^r}.$$

We define also the homogeneous norm

$$\|f\|_{\dot{B}^s_{p,r}} \coloneqq \left\|2^{qs}\left\|\dot{\Delta}_q f\right\|_{L^p}\right\|_{\ell^r(\mathbb{Z})}.$$

The two spaces H^s and $B^s_{2,2}$ are equal and we have

$$\frac{1}{C^{|s|+1}}\|u\|B^s_{2,2} \leq \|u\|H^s \leq C^{|s|+1}\|u\|B^s_{2,2}.$$

Remark 2.3.1

We have also the embedding

$$B^s_{p_1,r_1} \hookrightarrow B^{s+d\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}_{p_2,r_2}, \quad p_1 \leq p_2 \quad and \quad r_1 \leq, r_2.$$

Definition 2.3.2

Let $T > 0$ and $\rho \geq 1$, we denote by $L_T^\rho B_{p,r}^s$ the space of distribution f such that

$$\|f\|_{L_T^\rho B_{p,r}^s} := \left\| (2^{qs} \|\Delta_q f\|_{L^p})_{\ell^r} \right\|_{L_T^\rho} < +\infty.$$

Besides the usual mixed space $L_T^\rho B_{p,r}^s$, need Chemin-Lerner space $\tilde{L}_T^\rho B_{p,r}^s$ which defined as the set of all distributions f satisfying

$$\|f\|_{\tilde{L}_T^\rho B_{p,r}^s} := \left\| 2^{qs} \|\Delta_q f\|_{L_T^\rho L^p} \right\|_{\ell^r} < +\infty.$$

The relation between these spaces are detailed in the following lemma, which is a direct consequence of the Minkowski's inequality.

Lemma 2.3.1

If $s \in \mathbb{R}, \varepsilon > 0$ and $(\rho, p, r) \in [1, \infty]^3$, then we have

$$L_T^\rho B_{p,r}^s \hookrightarrow \tilde{L}_T^\rho B_{p,r}^s \hookrightarrow L_T^\rho B_{p,r}^{s-\varepsilon}\text{ , if } r \geq \rho \text{ and}$$

$$L_T^\rho B_{p,r}^{s+\varepsilon} \hookrightarrow \tilde{L}_T^\rho B_{p,r}^s \hookrightarrow L_T^\rho B_{p,r}^s\text{ , if } \rho \geq r.$$

2.4 Paradifferential calculus

In this section, we study the way that the product acts on Besov spaces see Chemin [2] and Bahouri [6].

Definition 2.4.1

We denote by $T_u v$ the following bilinear operator:

$$T_u v = \sum_q S_{q-1} u \, \Delta_q v.$$

The remainder of u and v denoted by $R(u, v)$ is given by the following bilinear operator:

$$R(u, v) = \sum_{|q-q'| \leq 1} \Delta_q u \, \Delta_{q'} v.$$

Just by looking at the definition, it is clear that

$$uv = T_u v + T_v u + R(u, v).$$

We need also to the following result [6] and [15], for a proof.

Lemma 2.4.1

Let $(p,a) \in [1,\infty]^2$, and v be a divergence free vector field of $\mathbb{R}^2$. Assume in addition that a and b such that $\frac{1}{a}+\frac{1}{b}=1$. Then we have

$$\sum_{q\in\mathbb{Z}} 2^{\frac{q}{2}} \left\| [\dot{\Delta}_q, v.\nabla]u \right\|_{L_t^1 L^p} \lesssim \|v\|_{\tilde{L}_t^a \dot{B}_{p,1}^{\frac{3}{4}}} \|u\|_{\tilde{L}_t^b \dot{B}_{p,1}^{\frac{3}{4}}} .$$

Moreover, we have for $s \in]-1,1[$,

$$\sum_{q\in\mathbb{Z}} 2^{qs} \left\| [\dot{\Delta}_q, v.\nabla]u \right\|_{L^p} \lesssim \|\nabla v\|_{L^\infty} \|u\|_{\dot{B}_{p,1}^s}.$$

The following see result is useful to prove the uniqueness of solution of our result, see [15], and we will give here the proof.

Proposition 2.4.1

Let v be a vector field with divergence $v=0$ and θ be any smooth function. Then there exists a constant $C>0$ such that

$$\|v.\nabla\theta\|_{\dot{B}_{\infty,1}^0} \le C\, \|v\|_{\dot{B}_{\infty,1}^0} \|\theta\|_{\dot{B}_{\infty,1}^1}$$

Proof

We decompose $v.\nabla\theta$ as:

$$v.\nabla\theta = T_v\theta + T_{\nabla\theta}v + R(v,\nabla\theta),$$

where,

$$T_v.\nabla\theta = \sum_{q\in\mathbb{Z}} \dot{S}_{q-1}v\nabla\dot{\Delta}_q\theta,$$

$$T_{\nabla\theta}v = \sum_{q} \dot{S}_{q-1}\nabla\theta\dot{\Delta}_q v,$$

and

$$R(v,\nabla\theta) = \sum_{\substack{q\in\mathbb{Z}\\ i\in\mathbb{Z}}} \dot{\Delta}_q v \dot{\Delta}_{q+i}\nabla\theta.$$

Therefore

$$\|v.\nabla\theta\|_{\dot{B}_{\infty,1}^0} \lesssim \|T_v.\nabla\theta\|_{\dot{B}_{\infty,1}^0} + \|T_{\nabla\theta}v\|_{\dot{B}_{\infty,1}^0} + \|R(v,\nabla\theta)\|_{\dot{B}_{\infty,1}^0}$$

$$:= \mathrm{I} + \mathrm{II} + \mathrm{III} \qquad (2.20)$$

For I , we have from the definition of Besov space $\dot{B}_{\infty,1}^0$ and Bernstein inequality, that

$$\|I\|_{\dot{B}_{\infty,1}^0} = \|T_v.\nabla\theta\|_{\dot{B}_{\infty,1}^0} \le \sum_{q\in\mathbb{Z}} \left\| \dot{S}_{q-1}v\nabla\dot{\Delta}_q\theta \right\|_{L^\infty}$$

$$\lesssim \sum_{q\in\mathbb{Z}} \left\|\dot{S}_q v\right\|_{L^\infty} \left\|\nabla\dot{\Delta}_q\theta\right\|_{L^\infty}$$

$$\lesssim \sum_{q\in\mathbb{Z}} \left\|\dot{S}_q v\right\|_{L^\infty} 2^q \left\|\dot{\Delta}_q\theta\right\|_{L^\infty}$$

$$\lesssim \|v\|_{\dot{B}^0_{\infty,1}} \|\theta\|_{\dot{B}^1_{\infty,1}} \qquad (2.21)$$

By the same way, we get for II, that is

$$\|II\|_{\dot{B}^0_{\infty,1}} = \sum_{q\in\mathbb{Z}} \left\|\dot{S}_{q-1}\nabla\theta\dot{\Delta}_q v\right\|_{\dot{B}^0_{\infty,1}}$$

$$\lesssim \sum_{q\in\mathbb{Z}} \left\|\dot{S}_{q-1}\nabla\theta\right\|_{L^\infty} \left\|\dot{\Delta}_q v\right\|_{L^\infty}$$

$$\lesssim \|\nabla\theta\|_{L^\infty} \|v\|_{\dot{B}^0_{\infty,1}}$$

$$\lesssim \sum_{q} \left\|\dot{\Delta}_q\nabla\theta\right\|_{L^\infty} \|v\|_{\dot{B}^0_{\infty,1}}.$$

Using again Bernstein inequality, we obtain

$$\|II\|_{\dot{B}^0_{\infty,1}} \lesssim \sum_{q} 2^q \left\|\dot{\Delta}_q\theta\right\|_{L^\infty} \|v\|_{\dot{B}^0_{\infty,1}}$$

$$\lesssim \|\theta\|_{\dot{B}^1_{\infty,1}} \|v\|_{\dot{B}^0_{\infty,1}} \qquad (2.22)$$

For the remainder term III, use Bernstein inequality again

$$\|III\|_{\dot{B}^0_{\infty,1}} = \|R(v,\nabla\theta)\|_{\dot{B}^0_{\infty,1}} \le \sum_{j\in\mathbb{Z}} \left\|\dot{\Delta}_j R(v,\nabla\theta)\right\|_{L^\infty}$$

$$\le \sum_{j\in\mathbb{Z}} \left\|\dot{\Delta}_j \left(\dot{\Delta}_q v\dot{\Delta}_{q+i}\nabla\theta\right)\right\|_{L^\infty}$$

$$\lesssim \sum_{\substack{q\ge j-3\\ i\in\{\mp1,0\}}} 2^j \left\|\dot{\Delta}_q v\right\|_{L^\infty} \left\|\dot{\Delta}_{q+i}\theta)\right\|_{L^\infty}$$

$$\lesssim \sum_{\substack{q\ge j-3\\ i\in\{\mp1,0\}}} 2^{j-q} \left\|\dot{\Delta}_q v\right\|_{L^\infty} 2^q \left\|\dot{\Delta}_{q+i}\theta)\right\|_{L^\infty}$$

$$\lesssim \|v\|_{\dot{B}^0_{\infty,1}} \|\theta\|_{\dot{B}^1_{\infty,1}} \qquad (2.23)$$

Combining now $(2.21), (2.22), (2.23)$, and (2.20), find

$$\|v.\nabla\theta\|_{\dot{B}^0_{\infty,1}} \lesssim \|v\|_{\dot{B}^0_{\infty,1}} \|\theta\|_{\dot{B}^1_{\infty,1}}$$

Now the proof of the proposition is complete.

Chapter III:
Around a transport-diffusion equations

3.1 Introduction

Transport equations arise in many mathematical problems and, in particular, in most partial differential equations related to fluid mechanics. Although the velocity field v and the source term g may depend (nonlinearly) on f, having a good theory for linear transport equations is an important first step for studying such partial differential equations.

This chapter is devoted to the study of the following class of transport equations

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta + |D|^\alpha \theta = f \\ \theta|_{t=0} = \theta_0, \end{cases} \qquad (TD)_\alpha$$

where θ_0, f and v stand for given initial data, external force, and vector field,

respectively. We aim to state some useful estimates for the dissipative term $|D|^\alpha$. We discuss also two kinds of estimates for $(TD)_\alpha$ as L^p estimate and smoothing effects.

3.2 Some estimation for the dissipative term $|D|^\alpha$

In this section, will give some useful estimates for any smooth solution of linear transport-diffusion model $(TD)_\alpha$. The proof of the following result can be found in [15].

Proposition 3.2.1

If $f \in \dot{B}^\alpha_{2,1}$ such that $\alpha \in [0,1[$, and let ψ be a Libshitz measure-preserving homeomorphism on $\mathbb{R}^d$. Then there exists a positive constant C_α, depend only on α, and such that

$$\| |D|^\alpha (f o\, \psi) - (|D|^\alpha f)\, o\, \psi \|_{L^2} \leq C e^{V(t)} V(t) 2^{\alpha q} \|f\|_{L^2},$$

with

$$V(t) \coloneqq \|\nabla v\|_{L^1_t L^\infty}.$$

Now, will prove the following result which describes the action of the semi group operator $e^{-t|D|^\alpha}$on distribution whose Fourier transform is supported in a ring, see [2] , [15] and [21].

Proposition 3.2.2

Let $p \in [1, +\infty]$ and (t, λ) any couple of positive real numbers. Suppose also that $\alpha \epsilon \mathbb{R}_+$ and v be a zero divergence, with $supp\, \mathcal{F}v$ in included in a ring C. Then there exists a positive constant c such that

$$\left\| e^{-t|D|^{\alpha}} v \right\|_{L^p} \leq c e^{-c^{-1} t \lambda^{\alpha}} \|v\|_{L^p}.$$

Proof

let φ in $D(\mathbb{R}^d \setminus \{0\}$, such that $\varphi \equiv 1$, near the ring C, then we can write

$$e^{t|D|^{\alpha}} v = \varphi\left(\frac{1}{\lambda}|D|\right) e^{t|D|^{\alpha}} v = \mathcal{F}^{-1}\left(\varphi\left(\frac{1}{\lambda}\xi\right) e^{-t|\xi|^{\alpha}} \hat{v}(\xi)\right)$$

$$= \mathcal{F}^{-1}(\varphi\left(\frac{1}{\lambda}\xi\right) e^{-t|\xi|^{\alpha}}) * \mathcal{F}^{-1}(\hat{v}(\xi))$$

$$:= g_{\lambda}(t, x) * v, \qquad (3.1)$$

where,

$$g_{\lambda}(t, x) = \mathcal{F}^{-1}\left(\varphi\left(\frac{1}{\lambda}\xi\right) e^{-t|\xi|^{\alpha}}\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi}. e^{-t|\xi|^{\alpha}} \varphi\left(\frac{1}{\lambda}\xi\right) d\xi$$

Taking the L^p of (3.1) get

$$\left\| e^{t|D|^{\alpha}} v \right\|_{L^p} = \|g_{\lambda}(t, x) * v\|_{L^p}.$$

Using lemma 1.5.3 (Young inequality for convolution) , we obtain that

$$\left\| e^{t|D|^{\alpha}} u \right\|_{L^p} \leq c \|g_{\lambda}\|_{L^1} \|v\|_{L^p} \qquad (3.2)$$

It remains then to estimate $\|g_{\lambda}\|_{L^1}$, for this purpose, since we have

$$g_{\lambda}(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi - t|\xi|^{\alpha}} \varphi\left(\frac{1}{\lambda}\xi\right) d\xi.$$

Taking the L^1 norm of both sides of last equality, and using lemma 1.5.5 (Cauchy Schwartz inequality), we get

$$\|g_{\lambda}\|_{L^1} \leq \int_{\mathbb{R}^d} |g_{\lambda}(x)|\, dx \leq \int_{\mathbb{R}^d} ((1 + |x|^2)^{-d} (1 + |x|^2)^d |g_{\lambda}(x)|) dx$$

$$\leq \|(1 + |x|^2)^{-d}\|_{L^1} \|(1 + |x|^2)^d g_{\lambda}\|_{L^\infty}$$

$$\leq C\|(1+|x|^2)^d g_\lambda(x)\|_{L^\infty} \qquad (3.3)$$

Thus, we have

$$\|(1+|x|^2)^d g(x)\|_{L^\infty} = sup_x|(1+|x|^2)^d g(x)| \qquad (3.4)$$

Since we have

$$g_\lambda(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi - t|\xi|^\alpha}\varphi\left(\frac{1}{\lambda}\xi\right)\, d\xi.$$

Let $x = \frac{1}{\lambda}\xi$, and we set

$$G_\lambda(t,x) \coloneqq \lambda^{-d} g_\lambda\left(t,\frac{x}{\lambda}\right) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi - t\lambda^\alpha|\xi|^\alpha}\varphi(\xi)\, d\xi\, g(t,x).$$

Thus

$$(1+|x|^2)^d G_\lambda(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(Id-\Delta_\xi\right)^d\left(\varphi(\xi)e^{-t\lambda^\alpha|\xi|^\alpha}\right)e^{ix\xi}d\xi.$$

Using now Leibnitz formula, yields

$$\left(Id-\Delta_\xi\right)^d\left(\varphi(\xi)e^{-t\lambda^\alpha|\xi|^\alpha}\right) = \sum_{\substack{\beta\leq\alpha \\ |\alpha\leq 2d|}} C_\beta^\alpha\, \partial^{\alpha-\beta}\,\varphi(\xi)\partial^\beta e^{-t\lambda^\alpha|\xi|^\alpha}.$$

Since φ is a supported in a ring, it does not contain a neighbourhood of zero, then we get for $\xi \in supp\ \varphi$, there exists a couple (c, C) of positive real numbers such that for any ξ in the support of φ,

$$\left|(\partial^\beta e^{-t\lambda^\alpha|\xi|^\alpha}\right| \leq C(1+t\lambda^\alpha)^{|\beta|}e^{-t\lambda^\alpha|\xi|^\alpha} \leq\ Ce^{-c^{-1}t\lambda^\alpha}.$$

Therefore,

$$\left|\left(Id-\Delta_\xi\right)^d\left(\varphi(\xi)e^{-t\lambda^\alpha|\xi|^\alpha}\right)\right| \leq\ Ce^{-c^{-1}t\lambda^\alpha}\sum_{\substack{\beta\leq\alpha \\ |\alpha\leq 2d|}} C_\beta^\alpha\left|\partial^{\alpha-\beta}\,\varphi(\xi)\right|$$

The term in the right hand side belongs to the space $L^1(\mathbb{R}^d)$, thus deduce that

$$(1+|x|^2)^d G_\lambda(x) \leq\ Ce^{-c^{-1}t\lambda^\alpha}.$$

This gives in (3.2) that is

$$\left\|e^{tD^\alpha}u\right\|_{L^p} \leq\ Ce^{-c^{-1}t\lambda^\alpha}\|v\|_{L^P}.$$

Now the proof of the proposition is complete.

3.3 Smoothing effects

Present here two kinds of estimates about a transport diffusion equation: L^p estimates and smoothing effects. The proof of the following L^p estimates can be found in [7], [11], and [21].

Proposition 3.3.1

Let $0 \leq \alpha \leq 2$ and v be a smooth divergence free vector field. Let also f be a smooth function and θ is a smooth solution of $(TD)_\alpha$. Then for every $p \in [1, \infty]$, we have

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau .$$

Proof:

We will prove the proposition for $p \geq 2$ only**.** The case $p \in [1,2[$, can be obtained by duality method. Then multiplying the first equation of (TD_α), by $|\theta|^{p-2}\theta$, we get

$$\int \partial_t \theta |\theta|^{p-2}\theta \, dx + \int |D|^\alpha \theta \, |\theta|^{p-2}\theta \, dx = \int f|\theta|^{p-2}\theta dx.$$

Integrating this last by parts, lead to

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + \int |\theta|^{p-2}\theta |D|^\alpha \theta dx = \int f|\theta|^{p-2}\theta dx \quad (3.5)$$

We use the following result which we can found in [15] , [25] and [27].

$$\int |D|^\alpha \theta \, |\theta|^{p-2}\theta dx \geq 0, \qquad (3.6)$$

and using Lemma 1.5.4 (Holder inequality) for the right hand side of (3.5), we get

$$\int f|\theta|^{p-2}\theta dx \leq \|f\|_{L^p}\|\theta\|_{L^p}^{p-1} \qquad (3.7)$$

Plugging (3.6) and (3.7) into (3.5), yields to

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p &\leq \frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + \int |\theta|^{p-2}\theta |D|^\alpha \theta dx. \\ &\leq \|f\|_{L^p}\|\theta\|_{L^p}^{p-1}\end{aligned}$$

It follows that

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p \le \|f\|_{L^p}\|\theta\|_{L^p}^{p-1}$$

Thus we have

$$\|\theta\|_{L^p}^{p-1}\frac{d}{dt}\|\theta\|_{L^p} \le \|f\|_{L^p}\|\theta\|_{L^p}^{p-1}$$

Dividing this last inequality by $\|\theta\|_{L^p}^{p-1}$, we get

$$\frac{d}{dt}\|\theta\|_{L^p} \le \|f\|_{L^p} \qquad (3.8)$$

Integrating in time the inequality (3.8), we get

$$\|\theta(t)\|_{L^p} - \|\theta_0\|_{L^p} \le \int_0^t \|f(\tau)\|_{L^p} d\tau\,.$$

Therefore,

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau\,.$$

We intend now to prove the following smoothing effects, which is the main result of this chapter and it is the main ingredient of our result in the next chapter.

Theorem 3.3.2

Let v be a smooth divergence free vector field of $\mathbb{R}^2$ such that $v \in L^1_{loc}(\mathbb{R}_+, Lip(\mathbb{R}^2)$ and $f \in L^1_{loc}(\mathbb{R}_+; \dot{B}^s_{2,1}), s > \frac{3}{2}$. We consider also a smooth solution θ of the transport-diffusion equation $(TD)_\alpha$, with $\theta_0 \in \dot{B}^s_{2,1}$. Then for every $r \in [1,\infty]$, there exists a positive constant c_s depend only on s and such that

$$\|\theta\|_{\tilde{L}^r_t \dot{B}^{s+\frac{1}{2r}}_{2,1}} \le c_s\, e^{cV(t)}\left(\|\theta_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_t \dot{B}^s_{2,1}}\right),$$

where,

$$V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

And if $v = \nabla^\perp |D|^{-1}\theta$, the above estimate is also valid for $s > -1$.

Remark 3.3.1

Note that the theorem is also true in the case of $s \in]-1,1[$, and $(p,r) \in [1,\infty]^2$ which is proved in [15].

Proof of Theorem 3.3.2

To prove our result, we use a new approach based on Lagrangian coordinates combined with paradifferential calculus. The idea of the proof will be done in the spirit of [13], [15], [17] and [22]. First, we prove the smoothing effects for a small interval of time depending of vector v, but it depends not on the initial data. In the second step, we proceed to division in time thereby extending the estimate at any time arbitrary chosen positive.

3.3.1 Local estimates

We divide the proof into two cases: case 1 is $r = \infty$, and case 2: $r < \infty$.

Case 1: $\boldsymbol{r} = \infty$.

We localise in frequency the evolution equation, and rewriting the equation in Lagrangian coordinates. Let $q \in \mathbb{N}$, then the Fourier localized function $\dot{\Delta}_q\theta = \Delta_q\theta, q \in \mathbb{N}$ satisfies

$$\Delta_q(\partial_t\theta) + \Delta_q(v.\nabla\theta) + \Delta_q|D|^{\frac{1}{2}}\theta = \Delta_q f.$$

Now using the notation $[\,\Delta_q\,, v.\nabla\,]\theta = \Delta_q(v.\nabla\theta) - v.\nabla\Delta_q\theta$, we get

$$\Delta_q(v.\nabla\theta) = [\,\Delta_q\,, v.\nabla\,]\theta + v.\nabla\,\Delta_q\theta.$$

gives that

$$\partial_t\Delta_q\theta + \, v.\nabla\,\Delta_q\theta + [\,\Delta_q\,, v.\nabla\,]\theta\Delta_q + |D|^{\frac{1}{2}}\,\Delta_q\theta = \Delta_q\, f.$$

Therefore

$$\partial_t\Delta_q\theta + \, v.\nabla\,\Delta_q\theta + |D|^{\frac{1}{2}}\,\Delta_q\theta = \Delta_q\, f - [\,\Delta_q\,, v.\nabla\,]\theta := F_q.$$

From Proposition 3.3.1, we have

$$\left\|\Delta_q\theta(t)\right\|_{L^2} \leq \left\|\Delta_q\theta_0\right\|_{L^2} + \int_0^t \left\|F_q(\tau)\right\|_{L^2} d\tau$$

Multiplying this last by 2^{qs} and summing over q, yields,

$$\sum_q 2^{qs}\left\|\Delta_q\theta(t)\right\|_{L^2} d\tau \leq \sum_q 2^{qs}\left\|\Delta_q\theta_0\right\|_{L^2} + \int_0^t \sum_q 2^{qs}\left\|F_q(\tau)\right\|_{L^2} d\tau.$$

This gives that

$$\|\theta\|_{\dot{B}^s_{2,1}} \leq \|\theta_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_t\dot{B}^s_{2,1}} + C\int_0^t \sum_q 2^{qs}\left\|[\,\Delta_q\,, v.\nabla\,]\theta(\tau)\right\|_{L^2} d\tau.$$

Therefore

$$\|\theta\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \leq \|\theta_0\|_{\dot{B}_{2,1}^s} + \|f\|_{L_t^1 \dot{B}_{2,1}^s} + \int_0^t \sum_q 2^{qs} \big\|[\,\Delta_q\,, v.\nabla\,]\theta(\tau)\big\|_{L^2}\, d\tau$$

By using Lemma 2.4.1

$$\|\theta\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \leq \|\theta_0\|_{\dot{B}_{2,1}^s} + \|f\|_{L_t^1 \dot{B}_{2,1}^s} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\theta\|_{\tilde{L}_\tau^\infty \dot{B}_{2,1}^s} d\tau\,.$$

Using Gronwall's Lemma 1.5.6, to obtain

$$\|\theta\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \leq C\left(\|\theta_0\|_{\dot{B}_{2,1}^s} + \|f\|_{L_t^1 \dot{B}_{2,1}^s}\right) e^{c\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau} \quad (3.9).$$

This completes the proof of the result in the case of $r = \infty$.

Case 2: $r < \infty$:

Let us now introduce the flow ψ_q of the regularised velocity v,

$$\psi_q(t,x) = x + \int_0^t v\left(\tau, \psi_q(\tau,x)\right) d\tau.$$

We set

$$\bar{\theta}_q(t,x) = \Delta_q \theta(t, \psi_q(t,x)) \text{ and } \bar{F}_q(t,x) = F_q\left(t, \psi_q(t,x)\right).$$

Then we have the equation,

$$\partial_t \bar{\theta}_q + |D|^{\frac{1}{2}} \bar{\theta}_q = \bar{F}_q + |D|^{\frac{1}{2}}\big(\theta_q \; o\; \psi_q\big) - \left(|D|^{\frac{1}{2}} \theta_q\right) o\psi_q := \bar{F}_q^1 \quad (3.10)$$

Since the flow preserves Lebesgue measure, then we obtain

$$\big\|\bar{F}_q\big\|_{L^2} \leq \big\|\Delta_q f\big\|_{L^2} + \big\|[\Delta_q, v.\nabla]\theta\big\|_{L^2} \qquad (3.11)$$

Using now Proposition 3.2.1, we find that for $q \in \mathbb{Z}$

$$\left\||D|^{\frac{1}{2}} (f o\, \psi) - \left(|D|^{\frac{1}{2}} f\right) o\, \psi\right\|_{L^2} \leq Ce^{CV(t)} V(t) 2^{\frac{q}{2}} \|f\|_{L^2} \qquad (3.12)$$

with

$$V(t) \coloneqq \|\nabla v\|_{L_t^1 L^\infty}\,,$$

Putting (3.11) and (3.12) into (3.10), we obtain

$$\big\|\bar{F}_q^1\big\|_{L^2} \leq \big\|\Delta_q f\big\|_{L^2} + \big\|[\Delta_q, v.\nabla]\theta\big\|_{L^2} + Ce^{CV(t)} V(t) 2^{\frac{q}{2}} \big\|\Delta_q \theta\big\|_{L^2}$$

Now using the notation $V(t) \leq e^{CV(t)}$, in Chapter I, we obtain

$$\big\|\bar{F}_q^1\big\|_{L^2} \leq \big\|\Delta_q f\big\|_{L^2} + \big\|[\Delta_q, v.\nabla]\theta\big\|_{L^2} + Ce^{CV(t)} 2^{\frac{q}{2}} \big\|\Delta_q \theta\big\|_{L^2} \quad (3.13)$$

again will localize in frequency the equation (3.10) through the operator Δ_j , $j \in \mathbb{Z}$

$$\partial_t \Delta_j\, \bar{\theta}_q + |D|^{\frac{1}{2}} \Delta_j\, \bar{\theta}_q = \Delta_j \bar{F}_q^1, \quad (3.14)$$

where

$$\Delta_j \bar{F}_q^1 := \Delta_j \bar{F}_q + \Delta_j\, \left(|D|^{\frac{1}{2}} (\theta_q\ o\ \psi_q) - \left(|D|^{\frac{1}{2}} \theta_q \right)\, o\psi_q \right).$$

Then from equation(3.14), we have

$$\Delta_j\, \bar{\theta}_q(t,x) = e^{-t|D|^{\frac{1}{2}}} \Delta_j\, \theta_q^0 + \int_0^t e^{-(t-\tau)|D|^{\frac{1}{2}}} \Delta_j \bar{F}_q(\tau) d\tau$$

$$+ \int_0^t e^{-(t-\tau)|D|^{\frac{1}{2}}} \Delta_j \left(|D|^{\frac{1}{2}} (\theta_q\ o\ \psi_q) - \left(|D|^{\frac{1}{2}} \theta_q \right)\, o\psi_q \right).$$

Proposition 3.3.1 yields to

$$\left\| \Delta_j \bar{\theta}_q(t) \right\|_{L^2} \le \left\| e^{-t|D|^{\frac{1}{2}}} \Delta_j \theta_q^0 \right\|_{L^2} + \int_0^t \left\| e^{-(t-\tau)|D|^{\frac{1}{2}}} \Delta_j \bar{F}_q(\tau) \right\|_{L^2} d\tau$$

$$+ \int_0^t \left\| e^{-(t-\tau)|D|^{\frac{1}{2}}} \Delta_j \left(|D|^{\frac{1}{2}} (\theta_q\ o\ \psi_q) - \left(|D|^{\frac{1}{2}} \theta_q \right)\, o\psi_q \right) \right\|_{L^2} d\tau.$$

Using Proposition 3.2.2, and (3.14), we find

$$\left\| \Delta_j \bar{\theta}_q(t) \right\|_{L^2} \le C e^{-ct2^{\frac{j}{2}}} \left\| \Delta_j \theta_q^0 \right\|_{L^2} + C \int_0^t e^{-c(t-\tau)2^{\frac{j}{2}}} \left\| \Delta_q f(\tau) \right\|_{L^2} d\tau$$

$$+ C\, e^{CV(t)} 2^{\frac{q}{2}} \int_0^t e^{-c(t-\tau)2^{\frac{j}{2}}} \left\| \Delta_q \theta(\tau) \right\|_{L^2} d\tau$$

$$+ C \int_0^t e^{-c(t-\tau)2^{\frac{j}{2}}} \left\| \left[\Delta_q, v.\nabla \right] \theta(\tau) \right\|_{L^2} d\tau.$$

Integrating this last estimate with respect to time t and using Lemme 1.5.2 (Young inequality**)**, we have for every $r \in [1,\infty]$,

$$\left\| \Delta_j \bar{\theta}_q \right\|_{L_t^r L^2} \le\ C 2^{-j/2r} \left(\left(1 - e^{-crt2^{\frac{j}{2}}} \right)^{\frac{1}{r}} \left\| \Delta_j \theta_q^0 \right\|_{L^2} +\ \left\| \Delta_q f \right\|_{L_t^1 L^2} \right) + C e^{CV(t)}\, 2^{\frac{q-j}{2}} \left\| \Delta_q \theta \right\|_{L_t^r L^2}$$

$$+ 2^{-j/2r} \int_0^t \left\| \left[\Delta_q, v.\nabla \right] \theta(\tau) \right\|_{L^2} d\tau \qquad (3.15)$$

Let $N \in \mathbb{N}$ be a fixed number that will be chosen later, and since the flow ψ preserves Lebesgue measure then we write

$$2^{q(s+1/2r)} \left\| \Delta_q \theta \right\|_{L_t^r L^2} = 2^{q(s+1/2r)} \left\| \bar{\theta}_q \right\|_{L_t^r L^2}$$

$$\le 2^{q(s+1/2r)} \left(\sum_{|j-q| \ge N} \left\| \Delta_j \bar{\theta}_q \right\|_{L_t^r L^2} + \sum_{|j-q| < N} \left\| \Delta_j \bar{\theta}_q \right\|_{L_t^r L^2} \right)$$

$$\coloneqq I_1 + I_2 \qquad (3.16)$$

If $j - q \geq N$, it follows by using Lemma 2.2.3 that,

$$\left\|\Delta_j \bar{\theta}_q\right\|_{L_t^r L^2} \leq \mathrm{C}\ 2^{-|j-q|} e^{\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau} \left\|\Delta_q \theta\right\|_{L_t^r L^2}$$

$$\leq C\ 2^{-|j-q|} e^{V(t)} \left\|\Delta_q \theta\right\|_{L_t^r L^2}.$$

Therefore, get

$$I_1 \leq C\ 2^{-N} e^{V(t)} 2^{q(s+1/2r)} \left\|\Delta_q \theta\right\|_{L_t^r L^2} \qquad (3.17)$$

For the term I_2, have

$$I_2 = 2^{q(s+1/2r)} \sum_{|j-q|<N} \left\|\Delta_j \bar{\theta}_q\right\|_{L_t^r L^2},$$

use (3.15), yields

$$I_2 \leq \mathrm{C}\left(1 - e^{-crt2^{\frac{q}{2}}}\right)^{\frac{1}{r}} 2^{qs} \left\|\Delta_q \theta_0\right\|_{L^2} + C 2^{\frac{N}{2r}} 2^{qs} \left\|\Delta_q f\right\|_{L_t^1 L^2}$$

$$+ C 2^{\frac{N}{2}} e^{CV(t)} 2^{q(s+1/2r)} \left\|\Delta_q \theta\right\|_{L_t^r L^2}$$

$$+ C 2^{N/2r} 2^{qs} \int_0^t \left\|\left[\Delta_q, v.\nabla\right]\theta(\tau)\right\|_{L^2} d\tau \qquad (3.18)$$

Plugging now (3.17), (3.18), into (3.16), obtain that

$$2^{q(s+1/2r)} \left\|\Delta_q \theta\right\|_{L_t^r L^2} \leq C\ 2^{-N} e^{V(t)} 2^{q(s+1/2r)} \left\|\Delta_q \theta\right\|_{L_t^r L^2}$$

$$+ \mathrm{C}\left(1 - e^{-crt2^{\frac{q}{2}}}\right)^{\frac{1}{r}} 2^{qs} \left\|\Delta_q \theta_0\right\|_{L^2}$$

$$+ C 2^{\frac{N}{2r}} 2^{qs} \left\|\Delta_q f\right\|_{L_t^1 L^2} + C\ 2^{\frac{N}{2}} e^{CV(t)} 2^{q(s+1/2r)} \left\|\Delta_q \theta\right\|_{L_t^r L^2}$$

$$+ C\ 2^{N/2r} 2^{qs} \int_0^t \left\|\left[\Delta_q, v.\nabla\right]\theta(\tau)\right\|_{L^2} d\tau.$$

We set now

$$H_q^r(t) \coloneqq 2^{q(s+1/2r)} \left\|\Delta_q \theta\right\|_{L_t^r L^2}.$$

Therefore

$$H_q^r(t) \leq C\ \left[2^{-N} e^{CV(t)} + 2^{\frac{N}{2}} e^{CV(t)}\right] H_q^r(t)$$

$$+ \mathrm{C}\left(1 - e^{-crt2^{\frac{q}{2}}}\right)^{\frac{1}{r}} 2^{qs} \left\|\Delta_q \theta^0\right\|_{L^2} + C 2^{\frac{N}{2r}} 2^{qs} \left\|\Delta_q f\right\|_{L_t^1 L^2}$$

$$+ C\ 2^{N/2r} 2^{qs} \int_0^t \left\|\left[\Delta_q, v.\nabla\right]\theta(\tau)\right\|_{L^2} d\tau.$$

Now, claim that, there exists two constants $N \in \mathbb{N}$ and C_0 such that if $V(t) \leq C_0$, then

$$2^{-N}e^{CV(t)} + 2^{\frac{N}{2}}\, e^{CV(t)} \leq \frac{1}{2C}.$$

To show this, we take first t such that $V(t) \leq 1$, which is possible since $\lim_{t\to 0^+} V(t) = 0$. Second, we choose N in order to get $2^{-\mathrm{N}}e^{C} \leq \frac{1}{4C}$. By taking again $V(t)$ sufficiently small, we obtain that

$$2^{\frac{N}{2}}\, e^{C} \leq \frac{1}{4C}.$$

Therefore, there exists two constants $N \in \mathbb{N}$ and C_0 such that if $V(t) \leq C_0$, then

$$2^{-N}e^{CV(t)} + 2^{\frac{N}{2}}\, e^{CV(t)} \leq \frac{1}{2C} \qquad (3.19)$$

Under this assumption $V(t) \leq C_0$, we obtain for $q \geq -1$,

$$H_q^r(t) \leq \mathrm{C}\left(1 - e^{-crt2^{\frac{q}{2}}}\right)^{\frac{1}{r}} 2^{qs}\left\|\Delta_q\theta_0\right\|_{L^2} + C2^{qs}\ \left\|\Delta_q f\right\|_{L_t^1 L^2}$$

$$+C2^{qs}\int_0^t\left\|\left[\Delta_q, v.\nabla\right]\theta(\tau)\right\|_{L^2} d\tau \qquad (3.20)$$

summing over q, and using Lemma 2.4.1, we find for $V(t) \leq C_0$,

$$\|\theta\|_{\tilde{L}_t^r \dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq C\|\theta_0\|_{\dot{B}_{2,1}^s} + C\|f\|_{L_t^1\dot{B}_{2,1}^s} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\theta(\tau)\|_{\dot{B}_{2,1}^s}\, d\tau.$$

Using Lemma 1.5.4 (Holder inequality), yields

$$\|\theta\|_{\tilde{L}_t^r \dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq C\left(\|\theta_0\|_{\dot{B}_{2,1}^s} + \|f\|_{L_t^1\dot{B}_{2,1}^s}\right) + CV(t)\|\theta\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \qquad (3.21)$$

Plugging (3.9) into (3.21), we find

$$\|\theta\|_{\tilde{L}_t^r \dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq C\left(\|\theta_0\|_{\dot{B}_{2,1}^s} + \|f\|_{L_t^1\dot{B}_{2,1}^s}\right) e^{c\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau} \qquad (3.22)$$

Therefore, the result is proved for small time.

3.3.2 Globalization

Let us now see how to extend this for an arbitrary positive time T. take a partition ($T_i)_{i=0}^N$ of the interval ,$[0,T]$, such that

$$\int_{T_i}^{T_{i+1}} \|\nabla v(\tau)\|_{L^\infty} d\tau \approx C_0, \forall\, i \in [0,N].$$

Reproducing the same argument of (3.22), we obtain

$$\|\theta\|_{\tilde{L}^r_{[T_i,T_{i+1}]}\dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq C\left(\|\theta(T_i)\|_{\dot{B}^s_{2,1}} + C\int_{T_i}^{T_{i+1}} \|f(\tau)\|_{\dot{B}^s_{2,1}} d\tau\right)\exp\left(\int_{T_i}^{T_{i+1}} \|\nabla v(\tau)\|_{L^\infty} d\tau\right)$$

Summing these estimates on $i = 1$, to $i = N - 1$, and using triangle inequality, gives

$$\|\theta\|_{\tilde{L}^r_T\dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq C\left(\sum_{i=0}^{N-1} \|\theta(T_i)\|_{\dot{B}^s_{2,1}} + C\int_0^T \|f(T)\|_{\dot{B}^s_{2,1}} dT\right)\exp\left(c\int_0^T \|\nabla v(\tau)\|_{L^\infty} d\tau\right)$$

From (3.22), we have

$$\|\theta\|_{\tilde{L}^r_T\dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq CN\left(\|\theta_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_T\dot{B}^s_{2,1}}\right)e^{CV(T)}.$$

It suffices to choose N such that $CN \approx V(t)$, then

$$\|\theta\|_{\tilde{L}^r_T\dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq C\,V(t)\left(\|\theta_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_T\dot{B}^s_{2,1}}\right)e^{CV(T)}.$$

Therefore, get

$$\|\theta\|_{\tilde{L}^r_T\dot{B}_{2,1}^{s+\frac{1}{2r}}} \leq Ce^{CV(T)}\left(\|\theta_0\|_{\dot{B}^s_{2,1}} + \|f\|_{L^1_T\dot{B}^s_{2,1}}\right).$$

This is the desired result, and the proof of the theorem is now achieved.

Chapter IV:
Global existence and uniqueness for solution of 2D guasi geostrophic equation

4.1 Introduction

The theory of global and uniqueness result for the quasi geostrophic equation, with small initial data, is proved by many numerous authors, and in a different functional spaces, refer to [12],[19] and [28]

4.2 Main result

In this chapter, we will study the system $(QG)_\alpha$ with $= \frac{1}{2}$**,** that we study the system

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta + |D|^{\frac{1}{2}} \theta = 0 \\ div\, v = 0 \\ \theta|_{t=0} = \theta_0. \end{cases} \qquad (QG)_{\frac{1}{2}}$$

We will prove the existence and uniqueness solution for $(QG)_{\frac{1}{2}}$ in the Besov space $B^s_{2,1}$, $s > \frac{3}{2}$, and finally, we combine it with the results of [14] and [15]. Our result reads as follows.

Theorem 4.2.1

Let $\theta_0 \in B^s_{2,1}, s > \frac{3}{2}$, then there exists $T > 0$ such that the $(QG)_{\frac{1}{2}}$ equation has a unique solution θ such that

$$\theta \in C([0,T]; B^s_{2,1}) \cap L^1_T \dot{B}^{s+\frac{1}{2}}_{2,1} .$$

In other words, there exists $\beta > 0$, such that $||\theta_0||_{\dot{B}^1_{\infty,1}} \leq \beta$, then than we $T = \infty$.

Proof

The proof of this theorem can be given in four steps

Step 1: A priori estimates

Step 2: Global existence

Steps 3: Local estimates

Step 4: Uniqueness.

For conciseness, we shall provide the a priori estimates supporting the claim of the theorem, and give a complete proofs of the uniqueness and local existence parts, while the proof of the existence part will be shortened, and briefly described.

4.2.1 A priori estimates

The important quantities to bound for all time are the L^∞ norm of the vorticity and the Lipschitz norm of the velocity. The main step for obtain a Lipschitz bound is given an L^∞ bound of the vorticity. We will prove three kinds of a priori estimates: the first one deals with some easy estimates that one can obtained by energy estimates. The second one is concerned with a global a priori estimate of the Lipschitz norm of the velocity, and the L^∞ norm of the vorticity. The last a priori estimates concerned with some strong estimates. We start then with the following which is a direct consequence of Proposition 3.3.1.

Proposition 4.2.1

Let θ be a smooth solution of $(QG)_{\frac{1}{2}}$, and $\theta_0 \in L^2$. Then we have

$$\|\theta\|_{L^2} \lesssim \|\theta_0\|_{L^2} \ .$$

Now we prove the following

Proposition 4.2.2

Let $\theta_0 \in \dot{B}^1_{\infty,1}$, and let ω be the vorticity of the velocity, with $\omega \coloneqq \nabla v$. Then then there exists two constants $C,\ \beta > 0$, such that if $\|\theta_0\|_{\dot{B}^1_{\infty,1}} \leq \beta$, then we have $\forall t \in \mathbb{R}_+$

$$\|\nabla v\|_{L^1_t L^\infty} + \|\omega(t)\|_{L^\infty} \leq C_0 e^{C_0 t},$$

with C_0 depends only on the norm of the initial data.

Proof

First, we use Holder and Bernstein inequalities, the embeddings $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, combined with the fact that Riesz transform maps continuously homogenous Besov space into itself, get

$$V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$

$$\leq t\ \|\nabla v\|_{L^\infty_t \dot{B}^0_{\infty,1}}$$

$$\leq t \|v\|_{L^\infty_t \dot{B}^0_{\infty,1}}$$

$$\leq t\, \|\theta\|_{L_t^\infty \dot{B}^1_{\infty,1}} \qquad (4.2.1)$$

Now using Lemma 2.3.1

$$\|\theta\|_{L_t^\infty \dot{B}^1_{\infty,1}} \leq C\|\theta\|_{\tilde{L}_t^\infty \dot{B}^1_{\infty,1}} \qquad (4.2.2)$$

Plugging now (4.2.2) into (4.2.1), we obtain that

$$V(t) \coloneqq \|\nabla v\|_{L_t^1 L^\infty} \leq C\, t\, \|\theta\|_{\tilde{L}_t^\infty \dot{B}^1_{\infty,1}} \qquad (4.2.3)$$

can now estimate $\|\omega(t)\|_{L^\infty}$, for this use the fact that

$$\|\omega(t)\|_{L^\infty} \lesssim \|\nabla v(t)\|_{L^\infty},$$

and the embeddings $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, combined with the fact that Riesz transform maps continuously homogenous Besov space into itself, we get in view of (4.2.2),

$$\begin{aligned}
\|\omega(t)\|_{L^\infty} &\lesssim \|\nabla v(t)\|_{L^\infty} \\
&\lesssim \|\nabla v(t)\|_{\dot{B}^0_{\infty,1}} \\
&\lesssim \|\theta(t)\|_{\dot{B}^1_{\infty,1}} \\
&\lesssim \|\theta\|_{L_t^\infty \dot{B}^1_{\infty,1}} \\
&\lesssim \|\theta\|_{\tilde{L}_t^\infty \dot{B}^1_{\infty,1}} \qquad (4.2.4)
\end{aligned}$$

From (4.2.3) and (4.2.4), we get

$$\|\nabla v\|_{L_t^1 L^\infty} + \|\omega(t)\|_{L^\infty} \leq C\, (t+1)\, \|\theta\|_{\tilde{L}_t^\infty \dot{B}^1_{\infty,1}}.$$

Now using Theorem 3.3.2, for $p = \infty$, we obtain

$$\|\nabla v\|_{L_t^1 L^\infty} + \|\omega(t)\|_{L^\infty} \leq \|\theta_0\|_{\dot{B}^1_{\infty,1}} e^{CV(t)}.$$

Since $\theta_0 \in \dot{B}^1_{\infty,1}$, then there exists a constant $\beta > 0$, such that

$$\|\theta_0\|_{\dot{B}^1_{\infty,1}} \leq \beta, \qquad (4.2.5)$$

And since the function V depends continuously in time and $V(0) = 0$, then we can deduce for small initial data that V does not blow up, and then there exists a constant $C > 0$, such that

$$V(t) \leq C\|\theta_0\|_{\dot{B}^1_{\infty,1}}, \quad \forall t \in \mathbb{R}_+ \qquad (4.2.6)$$

Therefore, we get

$$\|\nabla v\|_{L^1_t L^\infty} + \|\omega(t)\|_{L^\infty} \leq C\|\theta_0\|_{\dot{B}^1_{\infty,1}} \,(t+1)\, e^{C\,\|\theta_0\|_{\dot{B}^1_{\infty,1}}}.$$

This gives that

$$\|\nabla v\|_{L^1_t L^\infty} + \|\omega(t)\|_{L^\infty} \leq C_0 e^{C_0 t}.$$

The task is now to find a global estimates for stronger norms of the solution of $(QG)_{\frac{1}{2}}$.

Proposition 4.2.3

Let $\theta_0 \in \dot{B}^s_{2,1}$, with $s > \frac{3}{2}$, and θ be a smooth solution of $(QG)_{\frac{1}{2}}$. Then we have

$$\|\theta\|_{\tilde{L}^\infty_t \dot{B}^s_{2,1}} + \|\theta\|_{\tilde{L}^1_t \dot{B}^{s+\frac{1}{2}}_{2,1}} + \|v\|_{\tilde{L}^\infty_t \dot{B}^s_{2,1}} \lesssim \|\theta_0\|_{\dot{B}^s_{2,1}}$$

Proof

First, we apply Theorem 3.3.2, we get

$$\|\theta\|_{\tilde{L}^\infty_t \dot{B}^s_{2,1}} + \|\theta\|_{\tilde{L}^1_t \dot{B}^{s+\frac{1}{2}}_{2,1}} \lesssim \|\theta_0\|_{\dot{B}^s_{2,1}} \; e^{CV(t)} \qquad (4.2.7)$$

Now, to estimate $\|v\|_{\tilde{L}^\infty_t B^s_{2,1}}$, we can writ v as

$$v = \Delta_{-1} v + \sum_{q \geq 0} \Delta_q v.$$

Then

$$\|v\|_{\tilde{L}^\infty_t B^s_{2,1}} \lesssim \|\Delta_{-1} v\|_{L^\infty_t L^2} + \|v\|_{\tilde{L}^\infty_t \dot{B}^s_{2,1}}$$

Using again Theorem 3.3.2, yields

$$\|v\|_{\tilde{L}^\infty_t \dot{B}^s_{2,1}} \lesssim \|v\|_{L^\infty_t L^2} + \|\theta\|_{\tilde{L}^\infty_t \dot{B}^s_{2,1}}$$

$$\lesssim \|v\|_{L^\infty_t L^2} + \|\theta_0\|_{\dot{B}^s_{2,1}} \qquad (4.2.8)$$

Now, since $v = (-R_2\theta, R_1\theta) = \left(\frac{-\partial_2}{|D|}\theta, \frac{\partial_1}{|D|}\theta\right)$, then by the continuity of Riesz transform, we get

$$\|v\|_{L^\infty_t L^2} \lesssim \|\theta_0\|_{L^2} \qquad (4.2.9)$$

Putting (4.2.9) into (4.2.8), we get

$$\|v\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \lesssim \|\theta_0\|_{L^2} + \|\theta_0\|_{\dot{B}_{2,1}^s} \qquad (4.2.10)$$

Combining (4.2.7) , (4.2.10),and using proposition 4.2.2 we get

$$\|\theta\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} + \|\theta\|_{\tilde{L}_t^1 \dot{B}_{2,1}^{s+\frac{1}{2}}} + \|v\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \lesssim \|\theta_0\|_{\dot{B}_{2,1}^s}.$$

This is the desired result.

4.2.2 Global Existence

Let us now outline briefly the proof of the existence of global solution to $(QG)_{\frac{1}{2}}$. We construct a global solution. First, we smooth out initial data

$$\theta_0^n = S_n\theta_0.$$

By definition of the operator S_n, there is a radial function $\chi \in D(\mathbb{R}^2)$, such that

$$\theta_0^n = S_n\theta^0 = 2^{2n}\chi(2^n.) * \theta_0.$$

Then, we have

$$\begin{aligned}\|\theta_0^n\|_{L^2} &\le \|2^{2n}\chi(2^n.) * \theta_0\|_{L^2}\\ &\le \|2^{2n}\chi(2^n.)\|_{L^1} \|\theta_0\|_{L^2}\\ &\le \|\chi\|_{L^1} \|\theta_0\|_{L^2}\\ &\le C\, \|\theta_0\|_{L^2},\end{aligned}$$

and in the Besov space, we have

$$\begin{aligned}\|\theta_0^n\|_{\dot{B}_{2,1}^s} &\le \sum_{q\le n-1} \left\|\dot{\Delta}_q\theta_0\right\|_{\dot{B}_{2,1}^s}\\ &\le \sum_{|q-p|\le 1} 2^{ps} \sum_{q\le n-1} \left\|\dot{\Delta}_p\dot{\Delta}_q\theta_0\right\|_{L^2}\\ &\le \sum_p 2^{ps} \left\|\dot{\Delta}_p\theta_0\right\|_{L^2}\\ &\le C\, \|\theta_0\|_{\dot{B}_{2,1}^s}.\end{aligned}$$

The fact that, $div\ v_0^n = 0$, due to the incompressibility of the vector field v_0. Let us now consider the system,

$$\begin{cases} \partial_t \theta_n + v_n . \nabla \theta_n + |D|^{\frac{1}{2}} \theta_n = 0, \\ v_n = (-R_2 \theta_n, R_1 \theta_n), \\ \theta_0^n(0,x) = S_n \theta_0(x), \\ (\theta_0, v_0) = (0,0) \end{cases} \qquad (4.2.11)$$

The global existence of the solutions is governed by V_n, where

$$V_n(t) := \int_0^t \|\nabla v_n(\tau)\|_{L^\infty} d\tau.$$

Since the initial data are smooths, then can construct locally in time a unique solution (θ_n, v_n). This solution is globally defined since the Lipshitz norm of the velocity, does not blow up in finite time by Proposition 4.2.3. Once again from the a priori estimates, have

$$\|\theta_n\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} + \|\theta_n\|_{\tilde{L}_t^1 \dot{B}_{2,1}^{s+\frac{1}{2}}} + \|v_n\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} \lesssim \|\theta_0\|_{\dot{B}_{2,1}^s} \qquad (4.2.12)$$

The control is uniform with respect to the parameter n. Thus if follows that up to an extraction that (v_n, θ_n) is weakly convergent to (v, θ) belonging to

$$L_T^\infty \dot{B}_{2,1}^s \times L_T^\infty \dot{B}_{2,1}^s \cap \tilde{L}_T^1 \dot{B}_{2,1}^{s+\frac{1}{2}}.$$

Now, will prove that the (v_n, θ_n) is a Cauchy in $L_T^\infty L^2 \times L_T^1 L^2$.
Let $(n, n_1) \in \mathbb{N}^2, v_{n,n_1} = v_n - v_{n_1}$ and $\theta_{n,n_1} = \theta_n - \theta_{n_1}$, then aaccording to the estimate (4.2.12), get

$$\|v_{n,n_1}\|_{L_T^\infty L^2} + \|\theta_{n,n_1}\|_{L_T^1 L^2} \lesssim \|v_{n,0} - v_{n_1,0}\|_{L^2} + \|\theta_{n,0} - \theta_{n_1,0}\|_{L^2} \quad .$$

This show that (v_n, θ_n) is of a Cauchy in the space $L_T^\infty L^2 \times L_T^1 L^2$. Hence, it converges strongly to (v, θ). This allows us to pass to the limit in the system (4.2.11) and then we get that (v, θ) is a solution of $(QG)_{\frac{1}{2}}$.

The continuity in time of $\boldsymbol{\theta}$:

Let us now sketch the proof of the continuity in time of θ, that is $\theta \in C(\mathbb{R}_+, B_{2,1}^s)$. From the definition of Besov space, have for $N \in \mathbb{N},\ T > 0$ and for $t, t_1 \in \mathbb{R}_+$,

$$\|\theta(t) - \theta(t_1)\|_{B_{2,1}^s} \leq \sum_{q<N} 2^{qs} \|\Delta_q \theta(t) - \Delta_q \theta(t_1)\|_{L^2}$$

$$+\sum_{q\geq N} 2^{qs}\left\|\Delta_q\theta(t)-\Delta_q\theta(t_1)\right\|_{L^2}$$

$$\lesssim \sum_{q<N} 2^{qs}\left\|\Delta_q\theta(t)-\Delta_q\theta(t_1)\right\|_{L^2}$$

$$+\sum_{q\geq N} 2^{qs}\left(\left\|\Delta_q\theta(t)\right\|_{L^2}+\left\|\Delta_q\theta(t_1)\right\|_{L^2}\right).$$

Therefore, have

$$\|\theta(t)-\theta(t_1)\|_{B^s_{2,1}} \lesssim \sum_{q<N} 2^{qs}\left\|\Delta_q\theta(t)-\Delta_q\theta(t_1)\right\|_{L^2} + C\sum_{q\geq N} 2^{qs}\left\|\Delta_q\theta\right\|_{L^\infty_t L^2}$$

$$:= I_1 + I_2 \qquad (4.2.13)$$

For any $\varepsilon > 0$, then there exists a number N such that

$$I_2 := \sum_{q\geq N} 2^{qs}\left\|\Delta_q\theta\right\|_{L^\infty_t L^2} \leq \varepsilon.$$

For I_1, we use Taylor's formula, we have

$$\Delta_q\theta(t)-\Delta_q\theta(t_1) = (t-t_1)\int_0^1 \partial_t\Delta_q\theta(s)ds.$$

Taking the L^2 norm of the above equation, multiplying both sides by 2^{qs}, and summing over $q < N$, we get

$$\sum_{q<N} 2^{qs}\left\|\Delta_q\theta(t)-\Delta_q\theta(t_1)\right\|_{L^2} \leq |t-t_1|\sum_{q<N} 2^{qs}\left\|\partial_t\Delta_q\theta\right\|_{L^\infty_t L^2}$$

$$\lesssim |t-t_1|\sum_{q<N} 2^{qs}\, 2^{-q}2^{q}\left\|\partial_t\Delta_q\theta\right\|_{L^\infty_t L^2}.$$

Therefore,

$$\sum_{q<N} 2^{qs}\left\|\Delta_q\theta(t)-\Delta_q\theta(t_1)\right\|_{L^2} \lesssim |t-t_1|2^N\|\partial_t\theta\|_{L^\infty_t B^{s-1}_{2,1}} \qquad (4.2.14)$$

It remains now to estimate $\|\partial_t\theta\|_{L^\infty_t B^{s-1}_{2,1}}$. For this, use the equation of θ:

$$\partial_t\theta = -v.\nabla\theta - |D|^{\frac{1}{2}}\theta \qquad (4.2.15)$$

And will prove that $\partial_t\theta$ in the space $L_t^\infty B_{2,1}^{s-1}$. To do this, have from Remark 2.3.1 that,

$$|D|^{\frac{1}{2}}\theta \in B_{2,1}^{s-\frac{1}{2}} \hookrightarrow B_{2,1}^{s-1} \qquad (4.2.16)$$

From free divergence of the velocity and Bernstein inequality we get,

$$\|v.\nabla\theta\|_{B_{2,1}^{s-1}} := \sum_q 2^{q(s-1)}\left\|\Delta_q(v.\nabla\theta)\right\|_{L^2}$$

$$\lesssim \sum_q 2^{qs}\left\|\Delta_q(v\,\theta)\right\|_{L^2}$$

$$\lesssim \|v\,\theta\|_{B_{2,1}^s}.$$

Therefore, we obtain

$$\|v.\nabla\theta\|_{B_{2,1}^{s-1}} \lesssim \|v\,\theta\|_{B_{2,1}^s} \qquad (4.2.17)$$

Since the space $B_{2,1}^s$ is an algebra with $s > 1$, then

$$\|v\,\theta\|_{B_{2,1}^s} \lesssim \|v\|_{B_{2,1}^s}\|\theta\|_{B_{2,1}^s} \qquad (4.2.18)$$

Putting (4.2.18) into (4.2.17), we obtain

$$\|v.\nabla\theta\|_{B_{2,1}^{s-1}} \lesssim \|v\|_{B_{2,1}^s}\|\theta\|_{B_{2,1}^s} \qquad (4.2.19)$$

Combining (4.2.16), and (4.2.19), get $\partial_t\theta \in L_t^\infty B_{2,1}^{s-1}$. This gives in (4.2.14), that

$$\sum_{q<N} 2^{qs}\left\|\Delta_q\theta(t) - \Delta_q\theta(t_1)\right\|_{L^2} \leq C|t-t_1|\,2^N\|v\|_{B_{2,1}^s}\|\theta\|_{B_{2,1}^s}$$

Therefore, get in view of (4.2.13), and for any $\varepsilon > 0$,

$$\|\theta(t)-\theta(t_1)\|_{B_{2,1}^s} \lesssim C|t-t_1|\|v\|_{B_{2,1}^s}\|\theta\|_{B_{2,1}^s} + \varepsilon$$

Using Proposition 4.2.3, with $q \geq 0$, get

$$\|v\|_{B_{2,1}^s}\|\theta\|_{B_{2,1}^s} \leq \|\theta_0\|_{B_{2,1}^s}.$$

This gives that,

$$\|\theta(t)-\theta(t_1)\|_{B_{2,1}^s} \leq C_0.$$

This proves the continuity of θ.

4.2.3 Local existence

The local time existence depends on the control of $V(t) \coloneqq \|\nabla v\|_{L_t^1 L^\infty}$. We distinguish two cases: $s > \frac{3}{2}$ and $s = \frac{3}{2}$.

Case 1: $s > \frac{3}{2}$

There exists $a > 1$, such that $s > 2 - \frac{1}{2a}$. From the inequality,

$$V(t) \leq \int_0^t \|\theta(\tau)\|_{\dot{B}^1_{\infty,1}} d\tau$$

And by using Lemma 1.5.4 of Holder inequality, we get for $\frac{1}{a} + \frac{1}{b} = 1$,

$$V(t) \lesssim \|\theta\|_{L_t^1 \dot{B}^1_{\infty,1}}$$

$$\lesssim t^{\frac{1}{b}} \|\theta\|_{L_t^a \dot{B}^1_{\infty,1}}.$$

Theorem 3.3.2 and Remark 3.3.1, gives

$$V(t) \lesssim t^{\frac{1}{b}} \|\theta_0\|_{\dot{B}^{1-\frac{1}{2a}}_{\infty,1}} e^{CV(t)} \qquad (4.2.20)$$

Then we deduce that there exist $\beta > 0$, such that

$$t^{\frac{1}{b}} \|\theta_0\|_{\dot{B}^{1-\frac{1}{2a}}_{\infty,1}} \lesssim \beta \qquad (4.2.21)$$

This gives that

$$V(t) \leq C_0 \qquad (4.2.22)$$

Now, using again Lemma 2.3.1 and Theorem 3.3.2, with $q \geq 0$, yields

$$\|\theta\|_{L_t^\infty B^s_{2,1}} \leq \|\theta\|_{\tilde{L}_t^\infty B^s_{2,1}} + \|\theta\|_{\tilde{L}_t^1 B^2_{2,1}}$$

$$\leq C \|\theta_0\|_{B^s_{2,1}}$$

Therefore, from $(4.2.21)$, we have

$$t^{\frac{1}{b}} \lesssim \beta \|\theta_0\|^{-1}_{\dot{B}^{1-\frac{1}{2a}}_{\infty,1}}.$$

Then for every $t \in [0, T]$, we deduce that the time existence T is bounded below by

$$\|\theta_0\|^{-b}_{\dot{B}^{1-\frac{1}{2a}}_{\infty,1}} \leq T.$$

Case 2: $\boldsymbol{s = \frac{3}{2}}$

Procedd the same calculation as in (3.20), we have

$$\|\theta\|_{L^1_t \dot{B}^1_{\infty,1}} \lesssim \sum_{q\in\mathbb{Z}} \left(1 - e^{-ct2^{\frac{q}{2}}}\right) 2^{3\frac{q}{2}} \left\|\Delta_q \theta_0\right\|_{L^\infty}$$

$$+ \sum_{q\in\mathbb{Z}} 2^{\frac{3q}{2}} \left\|[\Delta_q, v.\nabla]\theta\right\|_{L^1_t L^\infty}$$

$$\coloneqq I_1 + I_2 \qquad (4.2.23)$$

Using Lemma 1.5.4 of Holder inequality and lemma 2.4.1, then have

$$I_2 \lesssim \|v\|_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}} \|\theta\|_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}}$$

$$\lesssim \|\theta\|^2_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}} \qquad (4.2.24)$$

Putting (4.2.24) into (4.2.23), get

$$\|\theta\|_{L^1_t \dot{B}^1_{\infty,1}} \lesssim \sum_{q\in\mathbb{Z}} \left(1 - e^{-ct2^{\frac{q}{2}}}\right) 2^{\frac{3q}{2}} \left\|\Delta_q \theta_0\right\|_{L^\infty} + \|\theta\|^2_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}} \qquad (4.2.25)$$

It remains now to estimate $\|\theta\|^2_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}}$. For this, have as before, that

$$\|\theta\|_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}} \lesssim \sum_{q\in\mathbb{Z}} \left(1 - e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}} 2^{\frac{q}{2}} \left\|\Delta_q \theta_0\right\|_{L^\infty} + \sum_{q\in\mathbb{Z}} 2^{\frac{q}{2}} \left\|[\Delta_q, v.\nabla]\theta\right\|_{L^1_t L^\infty}$$

Using (4.2.24), obtain

$$\|\theta\|_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}} \lesssim C_0 \sum_{q\in\mathbb{Z}} \left(1 - e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}} 2^{\frac{q}{2}} \left\|\Delta_q \theta_0\right\|_{L^\infty} + \|\theta\|^2_{\tilde{L}^2_t \dot{B}^{\frac{3}{4}}_{\infty,1}} \quad (4.2.26)$$

Since have, as $t \to 0^+$,

$$\sum_{q\in\mathbb{Z}}\left(1-e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}}2^{\frac{3q}{2}}\left\|\Delta_q\theta_0\right\|_{L^\infty}\to 0.$$

Consider now, C_1 be a sufficient small constant, and we define

$$T_1 := sup\left\{t>0,\sum_{q\in\mathbb{Z}}\left(1-e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}}2^{\frac{3q}{2}}\left\|\Delta_q\theta_0\right\|_{L^\infty}<C_1\right\},$$

Then we have with $t\le T_1$ and $V(t)\le C_0$,

$$\|\theta\|_{\tilde{L}_t^2\dot{B}_{\infty,1}^{\frac{3}{4}}}\lesssim\sum_{q\in\mathbb{Z}}\left(1-e^{-ct2^{\frac{q}{2}}}\right)2^{\frac{3q}{2}}\left\|\Delta_q\theta_0\right\|_{L^\infty}\qquad(4.2.27)$$

Plugging now (4.2.27) into (4.2.25), get

$$\begin{aligned}V(t)&\le C\|\theta\|_{L_t^1\dot{B}_{\infty,1}^1}\\&\lesssim\sum_{q\in\mathbb{Z}}\left(1-e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}}2^{\frac{3q}{2}}\left\|\Delta_q\theta_0\right\|_{L^\infty}\\&+C\left(\sum_{q\in\mathbb{Z}}\left(1-e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}}2^{\frac{3q}{2}}\left\|\Delta_q\theta_0\right\|_{L^\infty}\right)^2\qquad(4.2.28)\end{aligned}$$

Therefore,

$$V(t)\lesssim\left(\sum_{q\in\mathbb{Z}}\left(1-e^{-ct2^{\frac{q}{2}}}\right)^{\frac{1}{2}}2^{\frac{3q}{2}}\left\|\Delta_q\theta_0\right\|_{L^\infty}\right)^2.$$

This gives that for C_1 sufficient small,

$$V(t)\le C_0.$$

Then in view of theorem 3.3.2, we get

$$\|\theta\|_{\tilde{L}_T^\infty B_{2,1}^{\frac{3}{2}}}+\|\theta\|_{L_t^1B_{2,1}^2}\le C\,\|\theta_0\|_{B_{2,1}^{\frac{3}{2}}}.$$

4.2.4 Uniqueness

In this paragraph, will establish a uniqueness result for the system $(QG)_{\frac{1}{2}}$ in a largest space, say in the space

$$B_T := L^\infty_T \dot{B}^1_{\infty,1} \cap \ L^1_T \dot{B}^1_{\infty,1}.$$

Suppose that $\theta_1$and θ_2 are two solutions of $(QG)_{\frac{1}{2}}$ with the same initial data and belonging to the space B_T, for a fixed time $T > o$. Therefore, we have

$$\begin{cases} \partial_t \theta_1 + v_1.\nabla\theta_1 + |D|^\alpha \theta_1 = 0 \\ div\ v_1 = 0 \\ \theta_1|_{t=0} = \theta_0 \end{cases} \qquad (4.2.29)$$

and

$$\begin{cases} \partial_t \theta_2 + v_2.\nabla\theta_2 + |D|^\alpha \theta_2 = 0 \\ div\ v_2 = 0 \\ \theta_2|_{t=0} = \theta_0 \end{cases} \qquad (4.2.30)$$

Taking the difference between (4.2.29) and (4.2.30), we get

$$\begin{cases} \partial_t(\theta_1 - \theta_2) + v_1.\nabla\theta_1 - v_2.\nabla\theta_2 + |D|^\alpha(\theta_1 - \theta_2) = 0 \\ div\ v_1 - div\ v_2 = 0 \\ (\theta_1 - \theta_2)|_{t=0} = 0 \end{cases} \qquad (4.2.31)$$

We calculate $v_1.\nabla\theta_1 - v_2.\nabla\theta_2$ as follows: Since

$$v_1.\nabla(\theta_1 - \theta_2) + (v_1 - v_2)\nabla\theta_2 = v_1\nabla\,\theta_1 - v_1\nabla\,\theta_2 + v_1\nabla\theta_2 - v_2\nabla\theta_2$$

$$= v_1.\nabla\,\theta_1 - v_2\nabla\theta_2.$$

Therefore

$$v_1.\nabla\,\theta_1 - v_2.\nabla\theta_2 = v_1.\nabla(\,\theta_1 - \theta_2) + (v_2 - v_1).\nabla\theta_2.$$

We set

$$\Theta := \theta_1 - \theta_2 \quad \text{and} \quad V = v_1 - v_2.$$

Then, we obtain the equations

$$\begin{cases} \partial_t\Theta + v_1.\nabla\Theta + V.\nabla\theta_2 + |D|^\alpha\Theta = 0 \\ div\ V = 0 \\ \Theta_1|_{t=0} = \theta_0 \end{cases} \qquad (4.2.32)$$

Therefore Remark 3.3.1 allows us to applying theorem 3.3.2, we get

$$\|\Theta\|_{\dot{B}^0_{\infty,1}} \lesssim e^{\|\nabla v_1\|_{L^1_t L^\infty}} \|V.\nabla\theta_2\|_{L^1_t \dot{B}^0_{\infty,1}} \qquad (4.2.33)$$

Now, we use Proposition 2.4.1 we obtain

$$\|V.\nabla\theta_2\|_{\dot{B}^0_{\infty,1}} \lesssim \|V\|_{\dot{B}^0_{\infty,1}}\|\theta_2\|_{\dot{B}^1_{\infty,1}}.$$

The continuity of Riesz transforms in the homogeneous Besov space $\dot{B}^0_{\infty,1}$, implies that,

$$\|V.\nabla\theta_2\|_{\dot{B}^0_{\infty,1}} \lesssim \|\Theta\|_{\dot{B}^0_{\infty,1}}\|\theta_2\|_{\dot{B}^1_{\infty,1}}$$

Putting the last inequality into (4.2.33), yields

$$\|\Theta\|_{\dot{B}^0_{\infty,1}} \lesssim e^{\|\nabla v_1\|_{L^1_t L^\infty}} \int_o^t \|V.\nabla\theta_2(\tau)\|_{\dot{B}^0_{\infty,1}} d\tau$$

$$\lesssim e^{\|\nabla v_1\|_{L^1_t L^\infty}} \int_o^t \|\Theta(\tau)\|_{\dot{B}^0_{\infty,1}} \|\theta_2(\tau)\|_{\dot{B}^1_{\infty,1}} d\tau.$$

Using Lemma 1.5.6 of Gronwall's inequality, get

$$\|\Theta\|_{\dot{B}^0_{\infty,1}} \leq Ce^{\|\nabla v_1\|_{L^1_t L^\infty}} e^{\|\theta_2\|_{L^1_t \dot{B}^0_{\infty,1}}}.$$

Since $\theta_2 \in L^1_T \dot{B}^0_{\infty,1}$ and $\nabla v \in L^\infty_T L^\infty$.

This gives the uniqueness of the solution Which is $\theta_1 = \theta_2$.

Comparison between some results about $(\boldsymbol{QG})_{\boldsymbol{\alpha}}$

The result of global existence and uniqueness for solutions of $(QG)_{\boldsymbol{\alpha}}$ is obtained by many numbreous authors and in a different functionals spaces. Mention the paper of [8], [14] ,[15] and combining these results. since the authors in [14], are proved the result for

$$(\alpha, p, q) \in \left]0, \frac{1}{2}\right] \times [2, \infty[\times [1, \infty[$$

and for the initial date θ_0 in the Besove space that is $\theta_0 \in B^{1+\frac{2}{p}-2\alpha}_{p,q}$. The result is also proved by Chae and Lee in [4], but for $p = 2$, $q = 1$ and in the critical Besov space $B^{2-2\alpha}_{2,1}$.

Note that in this work, take $p = 2$ and $s > \frac{3}{2}$ as a special case of [15] and see that if we take $q = \infty$, then the result of [15] is the best result and more precise than [14], because they are proved the result in a large space and the embedding between two Besov spaces $B^{1+\frac{2}{p}-2\alpha}_{p,1} \hookrightarrow B^{1-\alpha}_{\infty,1}$, for $p < \infty$. This embedding gives that

$$\|\theta_0\|_{B^{1-\alpha}_{\infty,1}} \leq C\|\theta_0\|_{B^{1+\frac{2}{p}-2\alpha}_{p,1}} < \infty.$$

Conclusion

In this work, extend the global existence and uniqueness for $(QG)_\alpha$ to the supercritical case, that is when $\alpha < 1$. Precisely, we taking $\alpha = \frac{1}{2}$ in our work, and proved the global existence and uniqueness result in the Besov spaces $B^s_{2,1}, s > \frac{3}{2}$.

The existence and uniqueness of the solutions of 2D quasi-geostrophic equation, is obtained by using some a priori estimation based on the Lipschitz norm of the velocity, and some strong estimates on some Besov spaces.

Recommendation

In this research, the researcher recommends the following problems for future work:

(1) Studying the existence and uniqueness of the solutions for $(QG)_\alpha$ in the subcritical, in the critical cases and in some functional spaces;

(2) Studying the same problem in dimension three;

(3) Studying the Boussinesq system which coupling the equation of the velocity v, and the equation of the temperature θ in a different functional spaces.

الملخص

الهدف الرئيسي من هذه الأطروحة هو دراسة الحل العام لمعادلة شبه المغناطيسية ثنائية البعد. تعمل هذه المعادلة كنماذج ثنائية الأبعاد تنشأ في ديناميكيات السوائل الجيوفيزيائية.

ندرس في هذه الرسالة نظرية وجود ووحدانية حل المعادلة شبه المغناطيسية في بعدين وفي فضاء بزوف الدالي لمنظومة المعادلات الاتية:

$$\begin{cases} \partial_t \theta + v.\nabla\theta + |D|^{\frac{1}{2}}\theta = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty[\\ div\, v = 0, \\ \theta|_{t=0} = \theta_0 \end{cases}$$

هذه المشكلة نوقشت بواسطة العديد من العلماء و في فضاءات دالية مختلفة و تم استخدام بعض النتائج التي تقودنا الى دراسة هذه المشكلة. و من أهمها إيجاد تقدير معادلة الدوران التي ترتبط بمعادلة تفاضلية لمتجه السرعة v .

References

[1]. H. Abidi and T. Hmidi, On the global Well-Posedness of the critical Quasi-Geostrophic equation, Article in SIAM Journal on Mathematical Analysis · January 2008 DOI: 10.1137/070682319 ·

[2]. H. Bahouri, J.Y. Chemin, R. Danchin. Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen Wissenschaften, 343. Springer, Heidelberg, (2011).

[3]. J. T. Beale, T. Kato and A. Majda. Remarks on the Breakdown of smooth solutions for the 3-D Euler Equations.Commun. Math. Phys. 94, (1984), no. 1, p. 61–66 .

[4]. J.-M. Bony, Calcul symbolique et propagation des singularit´es pour les ´equations aux d´eriv´ees. partielles non lin´eaires, Annales de l'´ecole sup´erieure, 14 (1981), 209-246.

[5]. D. Chae and J. Lee, Global well-posedness in the supercritical dissipative quasi-geostrophic equations, Asymptot. Anal. 38 (2004), no. 3-4, 339–358.

[6] J.-Y.Chemin . Perfect incompressible fluids. Oxford University Press. Clarendon Press, 187-1998.

[7] J.-Y.Chemin. Théorème d'unicité pour le système de Navier-Stokes tridimensionnel. J. Anal. Math. 77(1999). no. 5, 27-50.

[8]. Chen, C. Miao, Z, Zhang, A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation, Communications in mathematical physics 271, 821-838 (2007).

[9]. P. Constantin, D. C´ordoba and J. Wu,On the critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J., 50 (2001), 97-107.

[10] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J.Math. Anal, 30 (1999), 937-948.

[11] Cordoba, A. and Cordoba. A maximum principal applied to Quasi geostrophic equations. Com. Math. Phys, 249, n. 3 (2004), 511–528.

[12] T. Dlotko. M. B. Kania and Chunyou Sun. Quasi-geostrophic equation in $\mathbb{R}^2$, Volume 259, Issue 2, 15 July 2015, Pages 531-561.

[13] Hmidi, T. Regularite holderienne des poches de tourbillon visqueuses. J. Math. Pure Appl., 9, 84, n 11 (2005), 1455–1495.

[14] T. Hmidi and S. Keraani, Global solutions of the super-critical 2D Q-G equation in Besov spaces, Advances in Mathematics 214 (2007) 618–638.

[15] T. Hmidi and S. Keraani, On the global solutions of the super-critical 2D quasi-geostrophic equation in besov spaces , Volume 214, Issue 2, 1 October 2007, Pages 618-638 .

[16] N. Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space. Comm. Math. Phus. 251 (2004), 365-376.

[17] N. Ju, On the two dimensional quasi-geostrophic equations, Indiana Univ. Math . J. 54 (2005), no. 3, 897–926.

[18] N. Ju, Global solutions to the two dimensional quasi-geostrophic equation with critical or super-critical dissipation Math. Ann. 334 (2006), no. 3, 627-642.

[19] J. H. Lacasce, Geophysical fluid dynamics, Joe Lacasce Department for Geosciences University of Oslo P.O.Box 1022 Blindern 0315 Oslo, Norway. November 23, 2015.

[20] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.

[21] S. Sulaiman, Etude qualitative de quelques equations d evolution nonlineaires. These en Univeriste de Rennes 1. France, (2012).

[22] S. Sulaiman, global existence and uniqueness for a non linear Boussinesq system in dimension two, Journal of Maths. Physics. 51, 9 (2010).

[23] S. Sulaiman, On the Bernstein inequality, University Bulletin – ISSUE No. 22- Vol. (2) – June- (2019).

[24] S. Sulaiman. On the global existence for the axisymmetric Euler-Boussinesq system in critical Besov spaces, Asympto. Analy. Journal. 77, (2005), 89–121,

[25] J.Wu, Solutions to the 2D quasi-geostrophic equations in Holder spaces, Nonlinear Analysis, 62 (2005), 579-594.

[26] J. Wu, Global solutions of the 2D dissipative quasi-geostrophic equations in Besov spaces, SIAM J. Math. Anal. 36 (2004/2005), 1014-1030.

[27] K. Yamazaki, Global well-posedness of transport equation with nonlocal velocity in Besov spaces with critical and supercrttical dissipation, 2Department of Mathematics, Oklahoma State University, 401 Mathematical Sciences, Stillwater (2011), OK 74078, USA, (405)744-7995.

[28]. Jia Yuan, On regularity criterion for the dissipative quasi-geostrophic equations, J. Math. Anal. Appl. 340 (2008) 334–339