Inviscid Limit of the 2d Euler-Stratified Systems with Vorticity in the LBMO Space

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Abstract

The global result of inviscid limit of the Navier-Stokes equations with datain $LBMO$ spaces and others results are obtained by Bernicot, Elgindi and Keraani in2016. In the present paper, we extend this result for the Navier-Stokes stratified systemwith vorticity in the same space $LBMO$.

Keywords *: generalised Boussinesq system, Navier-Stokes equation, LBMO spaces, initial data.*

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الملخص

نظرية وجود و وحدانية حل معادلات نافير ستوكس لمائع غير قابل للإنضغاط مع شروط ابتدائية في فضاء $LBMO$ نوقشت بواسطة العلماء بيرنيكو و الجندى و كيرانى سنة 2016. في هذا البحث سوف نوسع مفهوم هذه النظرية لمعادلات نافير ستوكس ستراتيفايد في الفضاء $LBMO$.

Introduction

Generalized Boussinesq equations for the incompressible fluid flows in $\mathbb{R}^2$ are of the form

$$(1.1)\qquad \begin{cases} \partial_t v_\nu + v_\nu.\nabla v_\nu - \nu\Delta v_\nu + \nabla p_\nu = \theta_\nu e_2, \\ \partial_t \theta_\nu + v_\nu.\nabla\theta_\nu - \Delta\theta_\nu = 0, \\ div\, v_\nu = 0, \quad (v_\nu, \theta_\nu)|_{t=0} = (v_0, \theta_0), \end{cases}$$

Here the vector field $v_\nu = (v_\nu^1, v_\nu^2), v_\nu^j = v_\nu^j(x,t), j = 1,2, (x,t) \in \mathbb{R}^2 \times [0,\infty)$ Stands for the velocity of the field, and it is assumed to be divergence-free, the scalar function $\theta_\nu(x,\mathrm{t})$ denotes the temperature and $p_\nu = p_\nu(x,t)$ is the scalar pressure. The parameter ν is the kinematic viscosity and the vector e_2 is given by $e_2 = (0,1)$ .Note that the system (1.1) concides with the classical incompressible Navier-Stokes equations when the initial temperature θ_0 is identically constant. It reads as follows

$$(1.2)\qquad \begin{cases} \partial_t v_\nu + v_\nu.\nabla v_\nu - \nu\Delta v_\nu + \nabla p_\nu = 0, \\ div\, v_\nu = 0 \\ v_\nu|_{t=0} = v_0 \end{cases}$$

The mathematical theory of the incompressible Navier-Stokes equations (1.2) was initiated by Leray in [10]. He proved the global existence of a weak global solution of the system (1.2) in the energy space by using a compactness method. Nevertheless, the uniqueness of these solutions is only known for two spatial dimensions. A few decades later, in [4], Fujita and Kato proved local well-posedness in the critical Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, by using a fixed point argument and taking advantage of the time decay of the heat semi flow. The global existence of these solutions is only proved for small initial data and the question for large data remains an outstanding open problem. For more discussion, we refer the reader to [8,9,13]. In [2], Bernicot and Keraani extended Yudovich's result to some class of initial vorticity in a Banach space which is strictly implicated between L^∞ and BMO.

Concerning the inviscid limit problem, that is the convergence of the viscous solutions $(v_\nu)_{\nu>0}$ to the solution of the incompressible Euler equation, we will restrict ourselves

to the discussion of the following results. In [1**1**], the author proved that for $v_0 \in H^s$ with

$s > \frac{5}{2}$; the solutions $(v_\nu)_{\nu>0}$ converges in L^2 norm to the unique solution v of the Euler system and the rate of convergence is of order ν. By using an elementary interpolation argument we deduce strong convergence in the Sobolev spaces $\mathrm{H}^\alpha, \forall \alpha < s$.We note that this result is local in time in space dimension three and global in space dimension two.

Recently, Masmoudi [12] proved strong convergence for the same space H^s of initial data, his proof is based on the use of a cut-off procedure. We mention that the inviscid limit problem in the context of axisymmetric flows was studied in [5]. In a recent paper of [1], the authors proved that the solution v_ν of the system (1.2) converges to the solution v of the Euler system in L^2 space with initial vorticity in $L^p \cap LBMO$, with $p \in [1,2)$.

Let us now turn to the Navier-Stokes stratified system (1.1) which has been intensively studied in recent decades, and for which there have been proved many results related to the global well-posedness problem. When the viscosity ν equal to zero, we obtain the so-called Euler stratified system, this system is given by,

$$(1.3) \qquad \begin{cases} \partial_t v + v.\nabla v + \nabla p = \theta e_2, \\ \partial_t \theta + v.\nabla \theta - \Delta \theta = 0, \\ div\, v = 0, \quad (v,\theta)|_{t=0} = (v_0,\theta_0), \end{cases}$$

In space dimension three, the inviscid limit was proved for example in [7], and [14]. In [15],

Samira A. Sulaiman proved a global well-posedness result of the system (1.3) with

a bounded vorticity in the *LBMO* spaces. We mention also the paper of Samira A. Sulaiman [16], that we have proved similar result but with any function depend only on the temperature.

In this paper, we extend the result of [1] to the Navier-Stokes stratified system. The main result of this paper is the following.

Theorem 1.1. Assume $p \in [1,2)$. Let $v_0 \in L^2(\mathbb{R}^2)$ be a divergence-free vector field of vorticity $w_0 \in L^p \cap LBMO$ and let $\theta_0 \in L^2 \cap L^1$ a real valued function. Let (v_ν,θ_ν) resp. (v,θ) the solution of the system (1.1) resp. the system (1.3). Then there exists a constant $C = C(v_0,\theta_0)$, such that for all $t \in [0,T]$, we have

$$\|v_\nu(t) - v(t)\|_{L^2(\mathbb{R}^2)} + \|\theta_\nu(t) - \theta(t)\|_{L^2(\mathbb{R}^2)} \leq (C_0 \nu\, t\, log(1+t))^{\frac{1}{2}\exp(1-e^{C_0 t})}$$

Remark 1.2. In [15], we proved the global existence and uniqueness for $2D$ Euler stratified

system with $w_0 \in L^p \cap LBMO$. The additional assumptions $v_0 \in L^2(\mathbb{R}^2)$ and $\theta_0 \in L^2(\mathbb{R}^2)$ are easily propagated and we get $(v, \theta) \in L^\infty L^2(\mathbb{R}^2) \times L^\infty L^2(\mathbb{R}^2)$.

Remark 1.3. From the proof, the rate of convergence in the L^2 space is of order $\frac{1}{2}$ at $t = 0$.

2. Some useful lemma

In this preliminary section, we introduce some basic notations and recall the definition of

the $LBMO$ space. We gives also some important results that will be used later.

- We denote by C any positive constant than will change from line to line and C_0 a real positive constant depending on the size of the initial data.

  We will use the following notations:

- For any positive constants A And B, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.
- For all set $D \subset \mathbb{R}^2$ and every integrable function f, we define $Avg_D(f)$ by the relation :

$$Avg_D(f) \coloneqq \frac{1}{|D|} \int_D f(x)dx.$$

We recall the functional space $LBMO$, introduced in [2] and [3].

Definition 2.1 The space $BMO(\mathbb{R}^d)$ of bounded mean oscillations is the set of locally integrable functions f such that :

$$\|f\|_{BMO} := sup_B Avg_B |f - Avg_B(f)| < \infty,$$

where B is a ball in $\mathbb{R}^2$ with $\sup B > 0$.

Definition 2.2. The $LBMO$ norm is defined by

$$\|f\|_{LBMO} \coloneqq \|f\|_{BMO} + sup_{B,C} \frac{|Avg_C(f) - Avg_B(f)|}{1 + ln\left(\frac{1 - ln\, r_C}{1 - ln\, r_B}\right)}$$

where the supremum is taken over all pairs of balls B and C in $\mathbb{R}^2$ with $0 < r_B < 1$ and 2C ⊂ B.

The following interpolation lemma will be used in the paper, see [1].

Lemma 2.1. There exists a constant $C > 0$ such that for every smooth function f and for

every $p \in [2, +\infty[$ the following estimate holds

$$\| f \|_{L^p} \leq Cp \|f\|_{L^2 \cap BMO}.$$

We will need also to the following lemma see [3].

Lemma 2.2. (Osgood Lemma) Let ρ be a measurable function from $[t_0, T]$ to $[0, a], \gamma$

a locally integrable function from $[t_0, T]$ to $\mathbb{R}_+$ and μ a continuous and nondecreasing

function from $[0, a]$ to $\mathbb{R}_+$. Assume that, for some nonnegative no decreasing continuous

c, the function ρ satisfies

$$\rho(\tau) \leq c(t) + \int_{t_0}^{t} \gamma(\tau)\mu\big(\rho(\tau)\big)d\tau.$$

Then

$$-\mathcal{M}\big(\rho(\tau)\big) + \mathcal{M}(\mathrm{c}(\mathrm{t})) \leq \int_{\mathrm{t}_0}^{\mathrm{t}} \gamma(\tau)\mathrm{d}\tau,$$

with

$$\mathcal{M}(x) = \int_{x}^{a} \frac{1}{\mu(r)} \mathrm{d}r.$$

The following proposition will be useful later, see [5] for a proof.

Proposition 2.1. Let (ω, θ) be a smooth solution of the following system

$$(2.1) \qquad \begin{cases} \partial_t \omega + v.\nabla\omega + \nabla p = \partial_1 \theta, \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0, \\ div\, v = 0, \quad (\omega,\theta)|_{t=0} = (\omega_0, \theta_0), \end{cases}$$

Let also $\theta_0 \in \mathrm{L}^1 \cap \mathrm{L}^\mathrm{p}$ and $\omega_0 \in \mathrm{L}^2 \cap \mathrm{L}^\mathrm{p}$ with $2 \leq \mathrm{p} \leq \infty$. Then for $\mathrm{t} \in \mathbb{R}_+$, we have

$$\|\omega(\mathrm{t})\|_{\mathrm{L}^\mathrm{p}} + \|\nabla\theta\|_{\mathrm{L}^1_\mathrm{t}\mathrm{L}^\mathrm{p}} \leq \mathrm{C}_0 \log^{2-\frac{2}{\mathrm{p}}}(1+\mathrm{t}).$$

3. Proof of Theorem 1.1

Taking the L^2 inner product of the first equation of (1.1) with v_ν and using the Holder inequality, we get

$$\|v_\nu(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla v_\nu(\tau)\|_{L^2}^2 d\tau \leq \|v_0\|_{L^2}^2 + \|\theta_0\|_{L^2} \|v_\nu\|_{L^1_t L^2}.$$

On the other hand, we have

$$\begin{aligned} \|v_\nu(t)\|_{L^2}^2 \leq \|v_\nu(t)\|_{L^2}^2 + 2\nu\|\nabla v_\nu\|_{L^1_t L^2}^2 &\leq \|v_0\|_{L^2}^2 + \|\theta_0\|_{L^2}\|v_\nu\|_{L^1_t L^2} \\ &\leq \|v_0\|_{L^2}^2 + C\,\|\theta_0\|_{L^2}^2 + C\|v_\nu\|_{L^1_t L^2}^2, \end{aligned}$$

where we have used the well-known Young inequality : For any a and b, we have ($ab \leq \frac{a^2}{2} + \frac{b^2}{2}$). Thus

$$\|v_\nu\|_{L_t^1 L^2}^2 = (\int_0^t \|v_\nu(\tau)\|_{L^2}\, d\tau)^2$$
$$\leq (t^{1/2} \|v_\nu\|_{L_t^2 L^2})^2$$

$$\leq t \|v_\nu\|_{L_t^2 L^2}^2 \leq t \int_0^t \|v_\nu(\tau)\|_{L^2}^2 d\tau.$$

This gives that,

$$\|v_\nu(t)\|_{L^2}^2 \leq C_0 + C\, t \int_0^t \|v_\nu(\tau)\|_{L^2}^2 d\tau,$$

where C_0 depend only on $\|v_0\|_{L^2}$ and $\|\theta_0\|_{L^2}$. Gronwall's inequality gives that

$$\|v_\nu(t)\|_{L^2}^2 \leq C_0 e^{C_0 t^2}.$$

The vorticity $\omega_\nu \coloneqq \partial_1 v_\nu^2 - \partial_2 v_\nu^1$ satisfies the equation,

$$(3.1) \qquad \begin{cases} \partial_t \omega_\nu + v_\nu . \nabla \omega_\nu - \nu\, \Delta \omega_\nu = \partial_1 \theta_\nu, \\ \partial_t \theta_\nu + v_\nu . \nabla \theta_\nu - \Delta \theta_\nu = 0, \\ div\, v_\nu = 0, \quad (\omega_\nu, v_\nu)|_{t=0} = (\omega_0, \theta_0), \end{cases}$$

Then the L^2 estimate of ω_ν yields that

$$\|\omega_\nu(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \int_0^t \|\partial_1 \theta_\nu(\tau)\|_{L^2} d\tau,$$
$$\leq \|\omega_0\|_{L^2} + \|\nabla \theta_\nu\|_{L_t^1 L^2}.$$

It remain then to estimate $\|\nabla\theta_\nu\|_{L^1_tL^2}$. For this purpose, we take the scalar product of the second equation of (3.1) with θ_ν in L^2 space. Then the incompressibility condition $div\, v_\nu$ leads to the following energy estimate :

$$\frac{1}{2}\frac{d}{dt}\|\theta_\nu(t)\|_{L^2}^2 + \|\nabla\theta_\nu(t)\|_{L^2}^2 = 0.$$

Then

$$\frac{d}{dt}\|\theta_\nu(t)\|_{L^2}^2 + 2\|\nabla\theta_\nu(t)\|_{L^2}^2 = 0.$$

Integrating in time this last differential equation, we get

$$\|\theta_\nu(t)\|_{L^2}^2 + 2\|\nabla\theta_\nu\|_{L^1_tL^2}^2 = \|\theta_0\|_{L^2}^2.$$

Therefore,

$$\|\nabla\theta_\nu\|_{L^1_tL^2}^2 \lesssim \|\theta_0\|_{L^2}^2.$$

This implies that

$$\|\nabla\theta_\nu\|_{L^1_tL^2} \leq \|\theta_0\|_{L^2}$$

This gives that

$$\|\omega_\nu(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2}. \tag{3.2}$$

Now, let $V_\nu := v_\nu - v$, $\Theta_\nu := \theta_\nu - \theta$, $P_\nu := p_\nu - p$ and $\Omega_\nu := \omega_\nu - \omega$, where ω_ν is the vorticity of v_ν and ω is the vorticity of v then we obtain the equations,

$$\begin{cases} \partial_t V_\nu + v_\nu.\nabla V_\nu - \nu\Delta V_\nu + \nabla P_\nu = -V_\nu.\nabla v + \nu\Delta v + \Theta_\nu e_2, \\ \partial_t\Theta_\nu + v_\nu.\nabla\Theta_\nu - \Delta\Theta_\nu = -V_\nu\nabla\theta, \\ div\, V_\nu = 0, \quad (V_\nu, \Theta_\nu)|_{t=0} = 0, \end{cases} \tag{3.3}$$

Taking the inner product of the first equation of (3.3) with V_ν gives

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|_{L^2}^2+\nu\|\nabla V_\nu(t)\|_{L^2}^2$$
$$\leq |< V_\nu.\nabla v, V_\nu >| + \nu |< \Delta v, V_\nu >| + |< \Theta_\nu, V_\nu >|$$

Then we have,

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|_{L^2}^2 \leq |< V_\nu.\nabla v, V_\nu >| + \nu\, \|\nabla v\|_{L^2}\|\nabla V_\nu\|_{L^2} + \|\Theta_\nu\|_{L^2}\|V_\nu\|_{L^2}$$
$$:= I + II + III. \tag{3.4}$$

For the second estimate II, we have from (3.2) and Proposition 2.1,

$$II \leq \nu\|\omega\|_{L^2}(\|\omega_\nu\|_{L^2}+\|\omega\|_{L^2}) \leq \nu C_0\log(1+t). \tag{3.5}$$

For the first estimate I, we use the H\"older inequality and the continuity of the Riesz transform, to get with $r \geq 2$ that

$$I \leq \int_{\mathbb{R}^2}|\nabla v(t,x)|\,|V_\nu(t,x)|^2dx \leq \|\nabla v\|_{L^r}\|V_\nu\|_{L^{2r_1}}^2$$

Using Lemma 2.1, we obtain

$$I \leq C\, r\|\nabla v\|_{L^2\cap BMO}\|V_\nu\|_{L^{2r_1}}^2.$$

By L^2- continuity of Riez operator on $L^2 \cap BMO$ and Theorem 1.1 in[15], , we get

$$I \leq r\, C_0e^{C_0t}\|V_\nu(t)\|_{L^{2r_1}}^2.$$

We use now Holder inequality and Biot-Savart law, we obtain

$$\|V_\nu(t)\|_{L^{2r_1}}^2 \lesssim \|V_\nu\|_{L^\infty}^{\frac{2}{r}}\|V_\nu\|_{L^2}^{2-\frac{2}{r}}$$
$$\lesssim \|\Omega_\nu\|_{L^p\cap L^3}^{\frac{2}{r}}\|V_\nu\|_{L^2}^{2-\frac{2}{r}}$$
$$\lesssim (\|\omega_\nu\|_{L^p\cap L^3}+\|\omega\|_{L^p\cap L^3})^{\frac{2}{r}}\|V_\nu\|_{L^2}^{2-\frac{2}{r}}.$$

Since ω and ω_ν are uniformly bounded on $L^p \cap L^3$, p ∈ [1,2), then we get to

$$(3.6) \qquad I \leq r\, C_0 e^{C_0 t} \|V_\nu\|_{L^2}^{2-\frac{2}{r}}.$$

For the last estimate III we first estimate $\|\Theta_\nu\|_{L^2}$ For this, we apply the maximum principle to the second equation of (3.3) we get

$$\|\Theta_\nu\|_{L^2} \leq \int_0^t \|V_\nu . \nabla\theta(\tau)\|_{L^2} d\tau$$

$$\leq \|V_\nu\|_{L_t^\infty L^2} \|\nabla\theta\|_{L_t^1 L^2}$$

$$(3.7) \qquad \leq C_0\, log^2(1+t) \|V_\nu\|_{L_t^\infty L^2},$$

where we have used Proposition **2.1**. Thus we get

$$(3.8) \qquad III \leq C_0 log^2(1+t) \|V_\nu\|_{L_t^\infty L^2}^2$$

Plugging together (3.5), (3.6)and (3.8) into (3.4), we find

$$(3.9) \qquad \frac{d}{dt} \|V_\nu\|_{L_t^\infty L^2}^2 \leq C_0\, \nu\, log(1+t) + C_0 \left(r\, e^{C_0 t} \|V_\nu\|_{L_t^\infty L^2}^{2-\frac{2}{r}} + log^2(1+t) \|V_\nu\|_{L_t^\infty L^2}^2 \right).$$

We take $f_\nu(t) := \|V_\nu\|_{L_t^\infty L^2}^2$ and define the maximal time T_ν as

$$T_\nu := max\{t \leq T, sup_{\tau \in (0,t]} f_\nu(\tau) \leq e^{-2}\},$$

with $T_\nu \leq T$ Then for every $t \in]0, T_\nu[$, we can take $r = -\log(f_\nu(t))$ to obtain

$$\frac{d}{dt} f_\nu(t) \leq C_0 \nu\, l\, og(1+t) + C_0 \big(-\, e^{C_0 t} f_\nu(t) \log(f_\nu(t)) + f_\nu(t) log^2(1+t) \big)$$

$$\leq C_0\, \nu\, log(1+t) + C_0 \big(\log(1+t) -\ e^{C_0 t} \big) f_\nu(t) \log\big(f_\nu(t)\big).$$

Integrating in time , we get

$$f_\nu(t) \leq C_0 \nu\, t\, l\, og(1+t) + C_0 \int_0^t \big(\log(1+\tau) -\ e^{C_0 \tau} \big) f_\nu(\tau) \log\big(f_\nu(\tau)\big) d\tau.$$

We assume that $C_0 \nu\, t\, l\, og(1+t) \leq 1$ and applying now Osgood Lemma 2.2, we find for every $t \leq T_\nu$,

$$-\log(\log(f_\nu(t)) + \log(-\log(C_0\nu\, t\;\; l\, og(1+t)))$$
$$\leq 1 - e^{C_0 t} + 2\, t \log(1+t).$$

This yields for all $t \leq T_\nu$, that

$$f_\nu(t) \leq \left(C_0\nu\;\; t\;\; l\, og(1+t)\right)^{\exp(1-e^{C_0 t})}.$$

If we assume that $\left(C_0\nu\, t\;\; l\, og(1+t)\right)^{\exp(1-e^{C_0 t})} \leq e^{-2}$, we get $T_\nu = T$. This gives finally

$$f_\nu(t) \leq \left(C_0\nu\, t\;\; l\, og(1+t)\right)^{\exp(1-e^{C_0 t})}, \quad \forall\, t \in [0,T],$$

with $C_0 = C_0(\|\omega_0\|_{L^p \cap LBMO})$. Therefore,

$$\|V_\nu\|_{L_t^\infty L^2} \leq \left(C_0\nu\, t\;\; l\, og(1+t)\right)^{\frac{1}{2}\exp(1-e^{C_0 t})}, \quad \forall\, t \in [0,T].$$

This gives in (3.7) that,

$$\|V_\nu\|_{L_t^\infty L^2} + \|\Theta_\nu\|_{L_t^\infty L^2} \leq \left(C_0\nu\, t\;\; l\, og(1+t)\right)^{\frac{1}{2}\exp(1-e^{C_0 t})}, \quad \forall\, t \in [0,T].$$

This completes the proof.

Conclusion

We have proved in space dimension two, the inviscid limit of the Euler stratified system with initial vorticity ω_0 in the $LBMO$ space .

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