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ON NEUTROSOPHIC PRE-APPROXIMATION SPACE

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Abstract :

The purpose of this article is to extend some more topics of topology in neutrosophic topology. We introduce the concept of neutrosophic pre-approximation space and its properties. Pre-open (pre-closed) sets are investigated. In addition, we define pre- lower (pre- upper) approximations of neutrosophic sets ; we define a new type of sets called neutrosophic pre- rough (pre-exact) sets and investigate some of their properties. We will develop five well-defined new regions and study the relationship between them, which will be useful in the study of GIS.

Keywords : **Neutrosophic Topology, Neutrosophic Pre-approximation Space, Neutrosophic Pre-open sets, Neutrosophic Pre-exact sets, Neutrosophic Pre rough sets.**

الملخص العربي: الغرض من هذه البحث هو توسيع بعض موضوعات الطوبولوجيا في الطوبولوجيا النيتروسوفيكية. نقدم مفهوم الفضاء التقريبي المسبق النيتروسوفيكي وخصائصه. تم تقديم مجموعات ما قبل الفتح ومجموعات ما قبل الغلق. بالإضافة إلى ذلك، نعرف التقريبات النيتروسوفيكية ما قبل السفلية (ما قبل العلوية) للمجموعات النيتروسوفيكية؛ نعرف ايضا نوعًا جديدًا من المجموعات يسمى بالمجموعات النيتروسوفيكية ما قبل السفلية (ما قبل العلوية) ودرسنا بعض خصائصها. سنقوم بتطوير خمس مناطق جديدة محددة جيدًا ودراسة العلاقة بينها، الأمر الذي سيكون مفيدًا في دراسة نظم المعلومات الجغرافية.

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1-Introduction:

The world around us is characterized by privacy, ambiguity and contradiction, which led to our lack of full knowledge of its facts and events, so researchers have turned to invent and develop new ideas to try to reach better accurate and clear results. Zadeh [1] introduced the concept of fuzzy sets (FS for short) in 1965. Where each element had a degree of membership truth (t). Atanassov [2] introduced the degree of nonmembership/falsehood (f) and defined the intuitionistic fuzzy sets

One of the interesting generalizations of the theory of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F. Smarandache [4, 5], which deals with the degree of indeterminacy/neutrality (i) as an independent component. Smarandache [6] defined the Neutrosophic set by three functions: Truth function, indeterminacy function and false function that are independently related. (T Truth, F -Falsehood, I- Indeterminacy). A neutrosophic set is a powerful tool for dealing with unspecified and inconsistent data. The features that characterized this theory have led to great success in various fields such as medical diagnosis, database, topology, image processing, and decision-making problem [7, 8, 9, 10, 11]. The fusion of neutrosophic sets with rough sets theory is an important research direction, there exists two fundamental combinations of rough sets and neutrosophic sets [13, 14, 15]. In other research, A. A Salama, S.A. Alblow [16] introduced the concept of neutrosophic topology. In [17] A. Salama, et, al are proposed a new mathematical model called neutrosophic crisp sets and neutrosophic crisp topological space., R. Dhavaseelan, Saied Jafari [18] are introduced Neutrosophic generalized closed sets.

Pre-approximation space [3] is a very important generalization of rough set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Moreover, it is a mathematical tool for machine learning, information sciences and expert systems and is successfully applied in data analysis and data mining. This space based on the class of pre-open sets and deals with general binary relations. It helps to get a new classification for the universe. The basic idea of pre-approximation space is based upon the approximation of sets by a pair of sets known as the pre-lower approximation and the pre-upper approximation of a set.

The accuracy of neutrosophic sets has also called on many researchers to apply them in geographical information systems (GIS) [19], In GIS there is a need to model spatial regions with indeterminate boundary and under indeterminacy. In this research the structure of some new neutrosophic sets will give high accuracy and specifically to some elements that were within the data of non-identification where five new known regions were developed well known. Therefore, neutrosophic pre-approximation space is a powerful mathematical tool to deal with incompleteness

In this paper, we introduce the concept of neutrosophic pre-approximation space and its properties. First, we review some basic notions related to pre-approximation space and neutrosophic sets after that, we construct the neutrosophic pre- lower (pre- upper) approximations of sets, In order to generate a new type of sets called neutrosophic pre- rough (pre-exact) sets and discuss some of their interesting properties.

2- Pre-approximation Space:

In the sense of Pawlak [12], knowledge consists of a family of classification patterns for the domain of interest. This view is general enough to cover various understandings of this concept in literature. The basic concept of RST is the notion of approximation space, which is an ordered pair (U, R) where U is a nonempty finite set called the universe and R is an equivalence relation.

The concept of pre-open sets, introduced and studied by Abdel Monsef [20] has been extensively applied in general topology by several authors, This concept has been used to create Pre-approximation space , which has been associated with general binary relations.

Definition 2.1[20]: A subset A of a topological space (X, τ) is called pre-open if $A \subset \operatorname{int}(\operatorname{cl}(A))$.

Definition 2.2 [3] : Let U be a finite nonempty universal. Then the pair (U, R_p) is called a pre-approximation space, where R_p is a general relation referred to as a subbase for topology, τ and used to generate a class of pre-open sets $P_O(U, \tau)$.

Remark 2.1.: In definition 2.2, we used the symbol R_p to avoid confusion with R, which refers to an equivalence relation.

Definition 2.3[3]: For any $A \subseteq U$, the pre-lower and pre-upper approximations of A with respect to (U, R_p) denoted by $\underline{R_p}(A)$ and $\overline{R_p}(A)$ arerespectively denoted as follows:
$\underline{R}_p(A) = \cup\{M \in Po(U, \tau); M \subseteq A\}, \overline{R}_p(A) = \cap\{N \in Pc(U, \tau); A \subseteq N\}$.

Definition 2.4 : Let (U, R_p) be a pre-approximation space. $A \subseteq U$. We have :

i-The pre-boundary region of A is denoted by $p.b(A)$ is given by: $p.b(A) = \overline{R}_p(A) - \underline{R}_p(A)$

ii- The pre-positive region of A is denoted by $P.Pos\ (A)$ and defined by $P.Pos\ (A) = \underline{R_p}(A)$.

iii- The pre-negative region of A is denoted by $P.neg(A)$ and defined by $P.neg(A) = U - \overline{R_p}(A)$.

iv-The pre-external boundary region of A is denoted by $p.ext.b(A)$ and defined by

$p.ext.b(A) = \overline{R}(A) - \underline{R_p}(A)$.

vi-The pre-internal boundary region A is denoted by $p.int.b(A)$ and defined by

$p.int.b(A) = \overline{R_p}(A) - \underline{R}(A)$.

Definition 2.5 [3] : let (U, R_p) be a pre-approximation space. $A \subseteq U$. We say that:

i- A is a pre-rough with respect to R_p if and only $\underline{R_p}(A) \neq \overline{R_p}(A)$, which equivalently $p.b(A) \neq \emptyset$.

ii- A is said to be pre-definable (pre- exact) if and only if $p.b(A) = \emptyset$.

3-Neutrosophic Philosophy:

This section provides a summary of the basic concepts of neutrosophic sets and neutrosophic topological space.

Definition 3.1 [6] : Let X be a non-empty set. A neutrosophic set (NS for short) $A \subseteq X$ is an object having the form: $A = \{< \mu_A(x), \sigma_A(x), \nu_A(x)) >, x \in X \}$, where

$\mu_A(x)$ represents the degree of membership function.

$\sigma_A(x)$ represents the degree of indeterminacy.

$\nu_A(x)$ represents the degree of non-membership function respectively of each element $x \in X$ to the set A.

Remark 3.1 : A neutrosophic set $A = \{< \mu_A(x), \sigma_A(x), \nu_A(x) >, x \in X\}$ can be expressed as an ordered triple $< \mu_A(x), \sigma_A(x), \nu_A(x) >$ in $]0^-, 1^+[$ in X. Where $0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) > \leq 1^+$ and $0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+$.

Remark 3.2 [5] : For simplicity, we use the symbol $< \mu_A(x), \sigma_A(x), \nu_A(x) >$ for neutrosophic set $A = \{< \mu_A(x), \sigma_A(x), \nu_A(x) >, x \in X >, x \in X\}$.

Since Smarandache provided the neutrosophic logic by the neutrosophic components T, I and F, which representing the membership, indeterminacy and non-membership values respectively, where $]0^-, 1^+[$ is a nonstandard unit interval. Thus, the neutrosophic set can be expressed in this context, as in the following definition.

Definition 3.2 [6] : Suppose X is a non-empty set. We express the neutrosophic set $A \subseteq X$ with a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A. We can be recognized as an ordered triad $< T_A(x), I_A(x), F_A(x) >$. Where $T_A: X \to]0^-, 1^+[$, $I_A: X \to]0^-, 1^+[$, $F_A: X \to]0^-, 1^+[$, and $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 3.3 [6] : Let X be a non-empty set. A neutrosophic $A \subseteq X$ is said to be null or empty neutrosophic set if $T_A(x) = 0$, $I_A(x) = 0$, $F_A(x) = 1$ for all $x \in X$. It is denoted by:

$0_N = \{< x, 0, 0, 1 >: x \in X\}$.

Definition 3.4 : A neutrosophic $A \subseteq X$ is said to be an absolute (universe) neutrosophic set if $T_A(x) = 1$, $I_A(x) = 1$, $F_A(x) = 0$ for all $x \in X$. It is denoted by $1_N = \{< x, 1, 1, 0 >: x \in X\}$.

Definition3.5 [5] : Let T, I, F be subsets of] $0^-, 1^+$ [, with $\sup T = t_{sup}$, $\inf T = t_{\inf}$, $\sup I = i_{sup}$, $\inf I = i_{\inf}$, $\sup F = f_{sup}$ and $\inf F = f_{\inf}$. So :

$n_{sup} = t_{sup} + i_{sup} + f_{sup} \leq 3^+$, $n_{\inf} = t_{\inf} + i_{\inf} + f_{\inf} \geq 0^-$.

Therefore, $0^- \leq \inf(n) \leq \sup(n) \leq 3^+$.

In the previous definition, T, I, and F are called neutrophilic components, where T represents the value of truth, while I represents the value of indeterminacy and F represents the value of falsehood respectively referring to neutrosophy, neutrosophic logic.

Let X be a non-empty set, $A \subseteq X$ and $x \in X$. We can analyze the element's belonging x to set A by the following method: it is t% true in the set, i% indeterminate (unknown if it is) in the set, and f% false, where t varies in T, i varies in I, f varies in F. For software engineering proposals, the classical unit interval [0, 1] is used.

Remark 3.3 : In the single valued neutrosophic logic (t, i, f), we observe the following :

1- When the three components are independent, the sum of the components has a value ranging between 0 and 3, which means that: $0 \leq t + i + f \leq 3$.

2- If two of the components are dependent, while the third is independent of them, then $0 \leq t + i + f \leq 2$.

3- When the three components are dependent, we get $0 \leq t + i + f \leq 1$.

4- If two or three components are independent. We have one of the following options:

i-(Sum <1) because one of them made a field for incomplete information.

ii-(Sum > 1) because the information is paraconsistent and contradictory.

iii-(Sum = 1) that is, the information is complete.

In the neutrosophic philosophy, one might say that, any neutrosophic element belongs to any set, due to the percentages of truth, indeterminacy, and falsity involved ranging from 0 to 1, or even less than 0 or greater than 1.

Example 3.1 : Let A be a neutrosophic set.

1- x (0.7, 0.1, 0.2) ∈A : means, with a probability of 70% x ∈ A, with a probability of 20% x ∉ A. As for the rest, we cannot report it.

2- y (0, 0, 1) ∈ A : means sure ∉ A.

3- z (0, 1, 0) ∈ A : This means that, we cannot determine whether the element z belongs to the set A or not.

Definition 3.6 [16] :

Let A and B be NSs of the form $A = < \mu_A(x), \sigma_A(x), \nu_A(x) : x \in X >$ and $B = < \mu_B(x), \sigma_B(x), \nu_B(x) : x \in X >$. Then:

i- $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$.

ii- $A^C = < \nu_A(x), \sigma_A(x), \mu_A(x) : x \in X >$.

iii- $A \cap B = < \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x) : x \in X >$.

iv- $A \cup B = < \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_A(x), \nu_A(x) \wedge \nu_B(x) : x \in X >$.

Definition 3.7 : Let X be a non-empty set, a neutrosophic set A with $\mu_A(x)$=1, $\sigma_A(x)$=1 and $\nu_A(x)$=1, is called normal neutrosophic set. In other words A is called normal if and only if $\max_{x\in X} \mu_A(x)$= $\max_{x\in X} \sigma_A(x)$= $\max_{x\in X} \nu_A(x)$=1.

Proposition 3.1 [16] : Let $A = < \mu_A(x), \sigma_A(x), \nu_A(x) >$ be a neutrosophic set on a set X. Then the following properties hold:

i- $A \cup 0_N = A$.

ii- $A \cup 1_N = 1_N$.

iii- $A \cap 0_N = 0_N$.

iv- $A \cap 1_N = A$.

Definition 3.8 [17]: Let X be a non-empty set and τN be the collection of neutrosophic subsets of X satisfying the following properties:

i- $0N, 1N \in \tau N$.

ii-$T1 \cap T2 \in \tau N$ for any $T1, T2 \in \tau N$.

iii-$\cup Ti \in \tau N$ for every $\{Ti : i \in j\} \subseteq \tau N$.

In this case, the space $(X, \tau N)$ is called a neutrosophic topological space (N-T-S). The element of τN are called neutrosophic open set (NO) and its complement is a neutrosophic closed set(NC).

Definition 3.9 : Let $(X, \tau N)$ be neutrosophic topological space and $A = < \mu_A(x), \sigma_A(x), \nu_A(x) >$ be a neutrosophic set in X.Then neutrosophic interior and neutrosophic closure of A are defined by: $N.int(A) = \cup\{M : M \text{ is a } N-O \text{ set}, M \subseteq A\}$,

$$N.cl(A) = \cap\{N : N \text{ is a } N-C \text{ set}, A \subseteq N\}.$$

Proposition 3.2: Let $(X, \tau N)$ be neutrosophic topological space, $A = < \mu_A(x), \sigma_A(x), \nu_A(x) >$ be a neutrosophic set in X.Then

i- A is neutrosophic open set $\Leftrightarrow A = N.int(A)$

ii- A is neutrosophic closed set $\Leftrightarrow A = N.cl(A)$.

4. Neutrosophic Pre-Approximation Space:

This section is divided into four parts. We first clarify the concept of neutrosophic pre -open and pre-closed sets. In the second part, we define pre- lower (pre- upper) approximations of neutrosophic set that are the basic component of neutrosophic pre-approximation space and study its most important properties. Finally, we are going to create a new type of sets called pre-rough sets and investigate some of its properties.

4.1- Neutrosophic pre -open and pre-closed sets:

Definition 4.1: Let A be a neutrosophic set of a neutrosophic topology. Then A is said to be:

i- neutrosophic pre-open set of X (NPO) if there exists a neutrosophic open set NO such that $NO \subseteq A \subseteq NO(Ncl\,(A))$.

ii -neutrosophic pre-closed sets of X (NPC) if there exists a NC set such that

$N.cl(NC) \subseteq A \subseteq NC$.

Theorem 4.1: Let $(X, \tau N)$ be a N-T-S and $A = <\mu_A(x), \sigma_A(x), \nu_A(x)>$ be a neutrosophic set in X.Then:

i- A is neutrosophicpre pre- open set if and only if $A \subseteq N\,int\,(Ncl\,(A))$.

ii- A is neutrosophicpre pre- closed set if and only if $Ncl\,(N\,int\,(A)) \subseteq A$.

Proof: i-Let A is a neutrosophic pre- open set in X. (i.e.) $NO \subseteq A \subseteq NO(Ncl\,(A))$ for some NO. But $NO \subseteq N\,int\,(A)$.Thus $NO(Ncl\,(A)) \subseteq N\,int\,(Ncl\,(A))$.
Hence$A \subseteq NO(Ncl\,(A)) \subseteq N\,int\,(Ncl\,(A))$. Therefore, $A \subseteq N\,int\,(Ncl\,(A))$.

Conversely. Suppose $A \subseteq N\,int\,(Ncl\,(A))$. We have to prove that A is neutrosophic pre- open set. We know that $NO = N\,int\,(A)$ and $NO \subseteq A \subseteq NO(Ncl\,(A))$.

Therefore, A is a neutrosophic pre- open set.

ii- Suppose that A be NPC set in X. Then $NC(N\,int\,(A)) \subseteq A \subseteq NC$ for some NS closed set NC

However, we have$Ncl\,(A) \subseteq NC$.

Conversely, Let $Ncl\,(N\,int\,(A)) \subseteq A$. Then $Ncl\,(A) = NC$, but$Ncl\,(N\,int\,(A)) \subseteq A \subseteq NC$.
Therefore, A is a NPC set.

Example 4.1 : Let$X = \{a;\ b;\ c\}$. Let $A = \{\langle 0.5, 0.5, 0.5\rangle, \langle 0.4, 0.4, 0.6\rangle, \langle 0.5,0.5,0.5\rangle\}$,
$B = \{\langle 0.5, 0.5, 0.5\rangle, \langle 0.55, 0.55, 0.55\rangle, \langle 0.5,0.5,0.5\rangle\}$,
$C = \{\langle 0.6, 0.6, 0.4\rangle, \langle 0.6, 0.6, 0.4\rangle, \langle 0.5,0.5,0.5\rangle$. Then $\tau N = \{0N, 1N, A, B, C\}$ is a N-T-S. The set $D = \{\langle 0.5, 0.5, 0.5\rangle, \langle 0.4, 0.4, 0.6\rangle, \langle 0.4,0.4,0.6\rangle$ is pre-open set.

Theorem 4.2 : In neutrosophic topology$(X, \tau N)$, $A \subseteq X$. Then :

i- A is a NPO set iff A^c is a NPC set.

ii- A is a NPC set iff $A^c$is a NPO set.

Proof : Obvious

Theorem 4.3 : Let $(X, \tau N)$ be any N-T-S. We have :

i-The union of two neutrosophic pre-open sets again a neutrosophic pre-open set.
ii-The intersection of two neutrosophic pre-closed sets is also a neutrosophic pre-closed set.

Proof : i-Let A , B be any two neutrosophic pre-open sets in $(X, \tau N)$.then

$$A \subseteq N\,int\,(Ncl\,(A)), \qquad B \subseteq N\,int\,(Ncl\,(B)).$$

$$\Rightarrow A \cup B \subseteq N\,int\,(Ncl\,(A)) \cup (N\,int\,(Ncl\,(B))).$$

$$\Rightarrow A \cup B \subseteq N\,int\,[Ncl\,(A)) \cup Ncl\,(B)].$$

$\Rightarrow A \cup B \subseteq N\,int\,[Ncl\,(A \cup B)]$.

Therefore, $A \cup B$ is a neutrosophic pre-open set in *X*.

ii- Let A , B be any two neutrosophic pre-open sets in $(X, \tau N)$. Then

$$Ncl\,(N\,int\,(A)) \subseteq A,\ Ncl\,(N\,int\,(B)) \subseteq B$$

$$\Rightarrow Ncl\,(N\,int\,(A)) \cap Ncl\,(N\,int\,(B)) \subseteq A \cap B.$$

$$\Rightarrow Ncl\,(N\,int\,(A) \cap N\,int\,(B)) \subseteq A \cap B.$$

$$\Rightarrow Ncl\,(N\,int\,(A \cap B)) \subseteq A \cap B.$$

Hence $A \cap B$ is neutrosophic pre-closed set.

Remark 4.1 : The intersection of any two NPO sets need not be a NPO set in X. Also, the union of any two NPC sets need not be a NPC set on $(X, \tau N)$ as shown by the following example.

Example 4.3 : Let $X = \{a, b\}$. Let *A* ={ ⟨0.4, 0.8, 0.9⟩ , ⟨0.7, 0.5, 0.3 ⟩},

B ={ ⟨0.5,0.8,0.6⟩ , ⟨ 0.8,0.4,0.3⟩ }, *C* ={⟨ 0.4,0.7,0.9 ⟩,⟨ 0.6,0.4,0.4 ⟩},

D ={⟨ 0.5, 0.7, 0.5⟩ , ⟨ 0.8, 0.4, 0.6⟩ }. Then

$\tau N = \{0N, 1N, A, B, C, D\}$ is neutrosophic topology on X. Consider

$M = \{\langle 0.6, 0.8, 0.6\rangle, \langle 1.0, 0.5, 0.5\rangle\}$, $N = \{\langle 1.0, 1.0, 0.3\rangle, \langle 0.7, 0.3, 0.6\rangle\}$ are NPO sets but $M \cap N$ is not NPO set.

Theorem 4.4 : Let $(X, \tau N)$ be a N-T-S. If $\{A_i\}_{i \in I} \subseteq X$ be a collection of NPO sets. Then

i-$\bigcup_{i \in I} A_i$ is neutrosophic pre-open set in $(X, \tau N)$.

ii-$\bigcap_{i \in I} B_i$ is neutrosophic pre-closed sets in $(X, \tau N)$

Proof : i-Let $\{A_i, i \in I\}$ be a family of NPO sets. By definition 3.1, for each $i \in I$, we have $NO_i \subseteq A_i \subseteq NO_i(Ncl\,(A))$.

$$\Rightarrow \bigcup_{i \in I} NO_i \subseteq \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} NO_i\,(Ncl\,(A)).$$

$\Rightarrow \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} Nint_i\,(Ncl\,(A))$. Therefore, $\bigcup_{i \in I} A_i$ is a NPO set.

ii- Let $\{B_i, i \in I\}$ be a family of NPC sets. By definition 3.1 and theorem3.1, for each $i \in I$, we have $NC_i(Nint(B_i) \subseteq B_i \subseteq NC_i$.Then

$$\Rightarrow \bigcap_{i \in I} NC_i(Nint(B_i) \subseteq \bigcap_{i \in I} B_i \subseteq \bigcap_{i \in I} NC_i)).$$

$$\Rightarrow \bigcap_{i \in I} NCl_i(Nint(B_i) \subseteq \bigcap_{i \in I} B_i$$

Therefore, $\bigcap_{i \in I} B_i$ is a neutrosophic pre-closed set.

Theorem 4.5: Let $(X, \tau N)$ be a N-T-S. Then:

i-Every NO set in the N-T-S is a NPO set.

ii-Every NC set in the N-T-S is a NPC set.

Proof: i- Let *A* be NO set in N-T-S. Then by proposition 3.1, $A = Nint\,(A)$.But $A \subseteq Ncl\,(A))$.Then $Nint\,(A) \subseteq Nint\,(N\,cl(A)) \Rightarrow A \subseteq Nint\,(N\,cl(A))$.

Hence A is a NPO set.

ii- let *A* be NC set in N-T-S. Then by proposition 3.1, $A = Ncl\,(A)$. But $Nint\,(A) \subseteq A$. Then $Ncl(Nint\,(A)) \subseteq N\,cl(A) \Rightarrow Ncl(Nint\,(A)) \subseteq A$ .Hence A is a NPC set.

4.2- Neutrosophic pre-lower and pre-upper approximations

Definition 4.2: Let $(X,\tau N)$ be a N-T-S and $A = <\mu_A(x), \sigma_A(x), \nu_A(x)>$ be a neutrosophic set in X. Then the neutrosophic pre-interior of $A(N.pint(A))$and neutrosophic pre-closure of A $(N.pcl(A))$ are defined as:

$N.pint(A) = \cup\{M, M \text{ is a } N-pOS \text{ in } X \text{ and } M \subseteq A\}$,

$N.pcl(A) = \cap\{L, N \text{ is a } N-pCS \text{ in } X \text{ and } A \subseteq L\}$ respectively.

Clearly, neutrosophic pre-interior of A $(N.pint(A))$ is the union of all neutrosophic pre-open sets of X contained in A. Therefore $N.pint(A)$ is the largest neutrosophic pre-open set over X which is contained in A.

Neutrosophic pre-closure of A $(N.pcl(A))$is the intersection of all neutrosophic pre-closed sets of X containing$A$. Then $N.pcl(A)$is the smallest neutrosophic pre-closed set over X which contains$A$.

Definition 4.3: Let X be a nonempty universe of discourse. For an arbitrary neutrosophic relation R over $X \times X$ the pair $(X_{\tau N}, R_p)$ is called neutrosophic pre-approximation space (N-P-S).

The class of all neutrosophic pre-open sets is denoted by $N.pOS(X_{\tau N}, R_p)$ and the class of all neutrosophic pre-closed sets is denoted by$N.pCS(X_{\tau N}, R_p)$.

Definition 4.4: For any $A \in (X_{\tau N}, R_p)$ the neutrosophic pre-lower and pre-upper approximations denoted by $N.\underline{R_p}(A)$ and $N.\overline{R_p}(A)$ are defined respectively by:

$$N.\underline{R_p}(A) = \cup\{M, M \text{ is a } N.pOS \text{ in } X \text{ and } M \subseteq A\},$$

$$N.\overline{R_p}(A) = \cap\{L, L \text{ is a } N.pCS \text{ in } X \text{ and } A \subseteq L\}.$$

Remark 4.2: it is easy to see that $N.\underline{R_p}(A)$ and $N.\overline{R_p}(A)$ are two neutrosophic sets in U ,thus the mapping$N.\underline{R_p}(A)$, $N.\overline{R_p}(A)$:N(U) $\rightarrow$ N(U) are, respectively is exactly $N.pint(A)$, $N.pcl(A)$ in the N.T.S.

Proposition 4.1: Let A , B be any two neutrosophic sets in $(X_{\tau N}, R_p)$.Then

i- A is neutrosophic pre-open set $\Leftrightarrow A = N.\underline{R_p}(A)$

ii- A is neutrosophic pre-closed set $\Leftrightarrow A = N.\overline{R_p}(A)$.

Theorem 4.6 : Let A , B be any two neutrosophic sets in neutrosophic pre-approximation space

$(X_{\tau N}, R_p)$. Then the neutrosophic pre- lower (pre-upper) approximation operators satisfy the following properties.

1. $N.\overline{R_p}(0N) = 0N$, $N.\overline{R_p}(1N) = 1N$.
2. $N.\underline{R_p}(A) \subseteq A \subseteq N.\overline{R_p}(A)$.
3. $A \subseteq B \Rightarrow N.\overline{R_p}(A) \subseteq N.\overline{R_p}(B)$.
4. $A \subseteq B \Rightarrow N.\underline{R_p}(A) \subseteq N.\underline{R_p}(B)$.

Proof : Let $x \in N.\underline{R_p}(A)$ which means that $x \in \cup\{M, M \text{ is a } N.pOS \text{ in } X \text{ and } M \subseteq A\}$. Then there exists $M_o \in N.pOS(X_{\tau N}, R_p)$ such that $x \in M_o \subseteq A$. Thus $x \in A$.

Hence $N.\underline{R_p}(A) \subseteq$ A.

Also, let $x \in A$ and by definition of $N.\overline{R_p}(A) = \cap \{ L, L \text{ is a } N.pCS \text{ in } X \text{ and } A \subseteq L \}$.

Since $x \in A$, then $x \in L$, for all $L \in N.pCS(X_{\tau N}, R_p)$. Hence $A \subseteq N.\overline{R_p}(A)$.

Theorem 4.7: Let A be any neutrosophic set in neutrosophic pre-approximation space $(X_{\tau N}, R_p)$. Then

i- $(N.\underline{R_p}(A))^c = N.\overline{R_p}(A)^c$.

ii- $(N.\overline{R_p}(A))^c = N.\underline{R_p}(A)^c$.

Proof: i- Let $A \subseteq X$ be a NS in N-T-S. We have

$N.\underline{R_p}(A) = \cup \{ M, M \text{ is a } N.pOS \text{ in } X \text{ and } M \subseteq A\}$, Then

$(N.\underline{R_p}(A))^c = (\cup \{ M, M \text{ is a } N.pOS \text{ in } X \text{ and } M \subseteq A\})^c$

$= \cap \{ M^c, M^c \text{ is a } N.pCS \text{ in } X \text{ and } A^c \subseteq N \}$.

Replacing M^c by L, we get $(N.\underline{R_p}(A))^c = \cap \{ N, N \text{ is a } N.pCS \text{ in } X \text{ and } A \subseteq L \}$.

Therefore $(N.\underline{R_p}(A))^c = N.\overline{R_p}(A)^c$.

Analogously (ii) can be proved.

4.3- Neutrosophic Pre-rough sets:

In this section, we define a new type of sets called neutrosophic pre- rough sets and investigate some of their properties.

Definition 4.5 : Let A is a neutrosophic set in neutrosophic pre-approximation space $(X_{\tau N}, R_p)$. Then we say that:

i- A is a neutrosophic pre-rough if $N.\underline{R_p}(A) \neq N.\overline{R_p}(A)$.

ii- A is a neutrosophic pre-exact (pre-definable) if $N.\underline{R_p}(A) = N.\overline{R_p}(A)$.

Definition 4.6 : Let A be any neutrosophic set in neutrosophic pre-approximation space $(X_{\tau N}, R_p)$. Then we define a neutrosophic pre-boundary of A as follows:

$$N.Pb(A) = N.\overline{R_p}(A) - N.\underline{R_p}(A).$$

Proposition 4.2 : Let A be any neutrosophic set in neutrosophic pre-approximation space $(X_{\tau N}, R_p)$. Then A is a neutrosophic pre-definable set iff $N.Pb(A) = \phi$.

Proof. Obvious.

Proposition 4.3: Let $(X_{\tau N}, R_p)$ be a neutrosophic pre-approximation space. Then

i- Every neutrosophic exact set in X is a neutrosophic pre- definable.

ii- Every neutrosophic pre-rough set in X is a neutrosophic rough.

Proof: i- Let $A = < \mu_A(x), \sigma_A(x), \nu_A(x) >$ be a neutrosophic definable set in X, then

$N.cl(A) = N.Int(A) = A$, but we have

$A = N.Int(A) \subseteq N.pint(A) \subseteq N.pcl(A) \subseteq N.cl(A) = A$.Thus

$N.pint(A) = N.pcl(A)$. Hence A is a neutrosophic pre- definable.

iii- Since we have $N.pb(A) \subseteq N.b(A)$, for any neutrosophic set A in X. If $N.pb(A) \neq \phi$ which means that A is a neutrosophic pre-rough. Then $N.b(A) \neq \phi$. Hence A is a neutrosophic rough. Whenever, if $N.b(A) = \phi$ means that A is a neutrosophic definable set, then $N.pb(A) = \phi$ thus X is a neutrosophic pre-exact.

Obviously $(N.\underline{R_p}(A), N.\overline{R_p}(A))$ is a definable neutrosophic set in the neutrosophic pre-approximation space$(X_{\tau N}, R_p)$.

Definition 4.7: Let $(X_{\tau N}, R_p)$ be a neutrosophic pre-approximation space. $A \subseteq X$.We are developing five new regions that are well definable:

i-The neutrosophic pre-boundary region of A is denoted by $N.p.b(A)$ is given by:

$$N.pb(A) = N.\overline{R_p}(A) - N.\underline{R_p}(A).$$

ii- The neutrosophic pre-positive region of A is denoted by $N.P.Pos\ (A)$ and defined by:

$$N.P.Pos\ (A) = N.\underline{R_p}(A)\ .$$

iii- The neutrosophic pre-negative region of A is denoted by $N.P.neg(A)$ and defined by:

$$N.P.neg(A) = U - N.\overline{R_p}(A).$$

iv-The neutrosophic pre-external boundary region of A is denoted by $N.p.ext.b(A)$ and defined by:

$$N.p.ext.b(A) = N.\overline{R}(A) - N.\underline{R_p}(A).$$

vi-The neutrosophic pre-internal boundary region A is denoted by $N.p.int.b(A)$ and defined by:

$N.p.int.b(A) = N.\overline{R_p}(A) - N.\underline{R}(A)$.

Theorem 4.8: Let A be any neutrosophic set in neutrosophic pre-approximation space (X_{τ_N}, R_p). Then the following properties hold:

i- $N.Pb(A) \subseteq N.b(A)$

ii- $N.p.ext.b(A) \subseteq N.b(A)$.

iii- $N.Pb(A) \subseteq N.p.int.b(A)$.

Proof: i- $N.Pb(A) = N.\overline{R_p}(A) - N.\underline{R_p}(A)$

$$\subseteq N.\overline{R_p}(A) - N.\underline{R}(A)$$

$$\subseteq N.\overline{R}(A) - N.\underline{R}(A) = N.b(A).$$

(ii and iii) are obvious.

Theorem 4.9 : Let A, B be two neutrosophic sets in neutrosophic pre-approximation space (X_{τ_N}, R_p). Then the following properties hold:

i- $N.neg(A) \subseteq N.P.neg(A)$

ii- $N.P.neg(A \cup B) \subseteq N.P.neg(A) \cup N.P.neg(A)$.

iii- $N.P.neg(A \cap B) \supseteq N.P.neg(A) \cap N.P.neg(A)$

Proof : i- Since $N.\overline{R_p}(A) \subseteq N.\overline{R}(A)$ then $U - N.\overline{R}(A) \subseteq U - N.\overline{R_p}(A)$ but $(N.neg(A) = U - N.\overline{R}(A) \subseteq U - N.\overline{R_p}(A) = N.P.neg(A)$. Therefore $N.neg(A) \subseteq N.P.neg(A)$.

ii- Since $N.\overline{R_p}(A \cup B) \supseteq N.\overline{R_p}(A) \cup N.\overline{R_p}(B)$.So

$U - N.\overline{R_p}(A \cup B) \subseteq U - (N.\overline{R_p}(A) \cup N.\overline{R_p}(B))$.

$\Rightarrow N.P.neg(A \cup B) = U - N.\overline{R_p}(A \cup B) \subseteq U - (N.\overline{R_p}(A) \cup N.\overline{R_p}(B))$.

$= (N.\overline{R_p}(A) \cup N.\overline{R_p}(B))^c = (N.\overline{R_p}(A))^c \cap (N.\overline{R_p}(B))^c$

$= (U - N.\overline{R_p}(A)) \cap (U - N.\overline{R_p}(B))$

$= N.P.neg(A) \cap N.P.neg(B) \subseteq N.P.neg(A) \cap N.P.neg(B)$.

iii-Since $N.\overline{R_p}(A \cap B) \subseteq N.\overline{R_p}(A) \cap N.\overline{R_p}(B)$. Then

$U - (N.\overline{R_p}(A) \cap N.\overline{R_p}(B)) \subseteq U - N.\overline{R_p}(A \cap B)$, but

$N.P.neg(A \cap B) = U - N.\overline{R_p}(A \cap B) \supseteq U - (N.\overline{R_p}(A) \cap N.\overline{R_p}(B))$

$\supseteq U - (N.\overline{R_p}(A) \cup N.\overline{R_p}(B))$

$= (U - N.\overline{R_p}(A)) \cap (U - N.\overline{R_p}(B)) = N.P.neg(A) \cap N.P.neg(A)$.

4.4-Properties of neutrosophic pre-rough sets

Theorem 4.10: Let A, B be two neutrosophic pre-rough sets in neutrosophic pre-approximation space $(X_{\tau N}, R_p)$. Then the following properties hold:

i- $N.\underline{R_p}(A \cap B) \subseteq N.\underline{R_p}(A) \cap N.\underline{R_p}(B)$.

ii- $N.\underline{R_p}(A \cup B) \supseteq N.\underline{R_p}(A) \cup N.\underline{R_p}(B)$.

iii- $N.\overline{R_p}(A \cap B) \subseteq N.\overline{R_p}(A) \cap N.\overline{R_p}(B)$.

iv- $N.\overline{R_p}(A \cup B) \supseteq N.\overline{R_p}(A) \cup N.\overline{R_p}(B)$.

Proof. i-Since $A \cap B \subseteq A$, $A \cap B \subseteq B$, then by Theorem 4.5, we have $N.\underline{R_p}(A \cap B) \subseteq N.\underline{R_p}(A)$ and $N.\underline{R_p}(A \cap B) \subseteq N.\underline{R_p}(B)$. Hence $N.\underline{R_p}(A \cap B) \subseteq N.\underline{R_p}(A) \cap N.\underline{R_p}(B)$.

ii-Since we have $A \subseteq A \cup B$ and $B \subseteq A \cup B$.Then $N.\underline{R_p}(A) \subseteq N.\underline{R_p}(A \cup B)$ and $N.\underline{R_p}(B) \subseteq N.\underline{R_p}(A \cup B)$ .Therefore $N.\underline{R_p}(A) \cup N.\underline{R_p}(B) \subseteq N.\underline{R_p}(A \cup B)$.

iii- Since $A \cap B \subseteq A$, $A \cap B \subseteq B$, then by Theorem 4.5, we have $N.\overline{R_p}(A \cap B) \subseteq N.\overline{R_p}(A)$ and

$N.\overline{R_p}(A \cap B) \subseteq N.\overline{R_p}(B)$. Hence $N.\underline{R_p}(A \cap B) \subseteq N.\underline{R_p}(A) \cap N.\underline{R_p}(B)$.

iv-Similar to i.

Remark4.3 : The necessary and sufficient condition for achieving equality in the previous theory is to be the sets A, B be two neutrosophic pre-definable sets.

Theorem4.11 : Let A, B be two neutrosophic pre- definable sets in neutrosophic pre-approximation space $(X_{\tau N}, R_p)$. Then the following properties hold:

i- $N.\underline{R_p}(A \cup B) = N.\underline{R_p}(A) \cup N.\underline{R_p}(B)$

ii- $N.\overline{R_p}(A \cap B) = N.\overline{R_p}(A) \cap N.\overline{R_p}(B)$.

Proof : i- The first part of the proof was proven in the previous theory in general**.** For the converse inclusion, let $x \in N.\underline{R_p}(A \cup B)$ that means,

$x \in \cup \{M, M \text{ is a } N-pOS \text{ in } X \text{ and } M \subseteq A \cup B\}$. Then there exists

$M_o \in N.pOS(X_{\tau N}, R_p)$ Such that $x \in M_o \subseteq A \cup B$. We distinguish three cases :

<u>Case</u> (i) If $M_o \subseteq A$, $x \in M_o$ and M_o is a neutrosophic pre-open set, then $x \in N.\underline{R_p}(A)$.

<u>Case</u> (ii) $M_o \cap A = \emptyset$, then $M_o \subseteq B$ and $x \in M_o$, thus $x \in N.\underline{R_p}(B)$

Case (iii) $M_o \cap A \neq \emptyset$. Since $x \in M_o$ and M_o is a neutrosophic pre-open set, then

$x \in N.p.cl(A)$, for every M_o which has the above condition, thus, $x \in N.\overline{R_p}$ (A), but A is neutrosophic pre- definable set then $x \in N.\underline{R_p}(B)$. Hence, in three cases $x \in N.\underline{R_p}(A) \cup N.\underline{R_p}(B)$. Therefore $N.\underline{R_p}(A \cup B) = N.\underline{R_p}(A) \cup N.\underline{R_p}(B)$.

iii- Similar to (i).

Conclusion :

Neutrosophic sets are well equipped to handle missing data. This article has taken a step forward in developing methods that can be used to describe and identify regions or data that have been ambiguous and indeterminate to reach more accurate results. With the neutrosophic pre-approximation space, we introduced neutrosophic pre-lower (pre-upper) approximations of neutrosophic sets. A new type of neutrosophic sets, which we have called neutrosophic pre-rough sets, are defined, we can use neutrosophic pre-rough sets in spatial data models. Five new well-defined regions were developed and the relationship between them was studied. The main contributions of the paper can be described as follows:

- Possible applications are listed after the definition of neutrosophic pre-rough sets. It is very useful in the study of GIS.
- We defined some new operators to describe things, and developed some neutrosophic regions.
- Possibility to profitably manipulate the spatial body in terms of neutrosophic pre-rough sets.

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