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THE MOORE-PENROSE GENERALIZED INVERSE AND GENERALIZED CRAMER RULES FOR SOLVING SINGULAR LINEAR ALGEBRAIC SYSTEMS

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ABSTRACT

In this paper, we consider singular linear systems of algebraic equations on the form $Ax = b$. We introduce formulas for the Moore- Penrose generalized inverse of A depend on the ranks of A, and based on the adgugate matrix. Using these formulas, we get generalized Cramer rules for the given systems.

Keywords: The Moore-Penrose generalized inverse, full ranks, singular linear systems, Cramer rules.

الملخص باللغة العربية

في هذا العمل، نعتبر الأنظمة الخطية الشاذة من المعادلات الجبرية على الصورة $Ax = b$. نحن نقدم صيغ لمعكوس مور-بنروز المعمم للمصفوفة A تعتمد على الرتب التامة للمصفوفة A و مؤسسة على مصفوفة المعاملات المرافقة. باستخدام هذه الصيغ نتحصل على قوانين كرامر المعممة لتلك الأنظمة المعطاة.

الكلمات المفتاحية : معكوس مور-بنروز المعمم، الرتب التامة، الأنظمة الخطية الشاذة، قواعد كرمر.

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1. INTRODUCTION

Consider the following singular linear system of algebraic equations

$$Ax = b; A \in \mathbb{C}^{m\times n}, x \in \mathbb{C}^n, b \in \mathbb{C}^m \tag{1.1}$$

where $A = (a_{ij});\ i = \overline{1,m}, j = \overline{1,n}, x = (x_1\ x_2 \cdots x_n)^T, b = (b_1\ b_2 \cdots b_m)$. If $A \in \mathbb{C}^{n\times n}$ and is invertible (non-singular), then the system (1.1) is easy to solve, the unique solution is $x = A^{-1}b$. The unique solution for (1.1) using Cramer's rule when A is invertible is

$$x_j = \frac{|A_j|}{|A|}, j = \overline{1,n} \tag{1.2}$$

where $|A_j|$ denotes the determinant of A_j, A_j is the matrix obtained by replacing its jth column with the vector b. If A is an arbitrary matrix in $\mathbb{C}^{m\times n}$, then the system (1.1) becomes more difficult to solve. One of the methods to solve (1.1) is using the Moore-Penrose generalized inverse (or shortly MPGI**)** of A, denoted by $A^\dagger$.

There are representations of MPGI for matrices depend on full column rank, full row rank, and

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full rank factorization for the matrices [4]. If A has full column rank, then A has left inverse $A^\dagger$ satisfies

$$A^\dagger A \simeq I, AA^\dagger \not\simeq I \tag{1.3}$$

it is

$$A^\dagger = (A^*A)^{-1}A^*. \tag{1.4}$$

If A has full row rank, then A has right inverse $A^\dagger$ satisfies

$$AA^\dagger \simeq I, A^\dagger A \not\simeq I, \tag{1.5}$$

it is

$$A^\dagger = A^*(AA^*)^{-1}. \tag{1.6}$$

If A has full rank factorization, $A = BC$ then

$$A^\dagger = C^*(CC^*)^{-1}(B^*B)^{-1}B^*. \tag{1.7}$$

In this paper, the representations of MPGI for a matrix A in (1.1) using the adjugate matrix are given. These representations depend on full column rank, full row rank, full rank, and full rank factorization, we give some lemmas and theorems about that. Then we use these representations to get generalized Cramer rules for finding the least squares solution with the minimum norm for (1.1) and give theorems for that. Finally, we give numerical examples to illustrate our results.

In 1989, Wang [11] gave a Cramer's rule for finding the solution for a class of singular equations (1.1) when A belongs to the range of A^k, where k is the index of A.

In 2008, Kyrchei [5] gave Cramer's rule for quaternion systems of linear equations. In the same year, he [6] introduced analogues of the adjoint matrix for generalized inverses and obtained corresponding Cramer rules in some cases for some systems. In 2015, he [9] introduced Cramer's rule for some generalized inverses, and obtained the least squares solution with the minimum norm for (1.1) when $rank(A) = n$ and when $rank(A) = r \leq m < n$, also he obtained the least squares solution with the minimum norm for the system $xA = b$ for some cases, and other results for other systems.

Throughout this paper, $\mathbb{C}^{m\times n}$ is the set of m by n matrices with complex entries, $\mathbb{C}_r^{m\times n}$ be a subset of $\mathbb{C}^{m\times n}$ in which any matrix has rank r, I be the identity matrix, A^T be the transpose of a matrix A, and A^* be the conjugate transpose of A. The entries of A^* are a_{ij}^*, where $a_{i.}^*$ and $a_{.j}^*$ denote the ith row and jth column respectively. $(A)_{i.}b$, $(A)_{.j}b$ denote the matrices obtained from A by replacing its i row and j column with the vector b respectively. Let A_{ij} denotes the cofactor of A^*A, it is $(-1)^{i+j}$ times the determinant of the (i, j) minor of A^*A.

This paper is organized as follows. Some preliminaries are given in section 2. Section 3 gives representations of MPGI using the adjugate matrix dependence on the full column rank, full row rank, full rank, and full rank factorization. In section 4, we get the generalized Cramer rules using the representations given in section 3. In section 5, we give examples to illustrate our results.

2. PRELIMINARIES

In this section, we introduce some important definitions, algorithms, and theorems (we refer the reader to [2-4]).

Definition 2.1 (MPGI of a matrix)

If $A \in \mathbb{C}^{m\times n}$, then $A^\dagger \in \mathbb{C}^{n\times m}$ is unique and it is called the Moore-Penrose generalized inverse of A if it satisfies the following conditions:

(1) $AA^\dagger A = A$,

(2) $A^\dagger AA^\dagger = A^\dagger$,

(3) $(AA^\dagger)^* = AA^\dagger$,

(4) $(A^\dagger A)^* = A^\dagger A$.

Definition 2.2 (The row echelon form)

A matrix $E \in \mathbb{C}^{m\times n}$ which has rank r is said to be in row echelon form if E is of the form

$$E = \begin{bmatrix} D_{r\times n} \\ \cdots \\ O_{(m-r)\times n} \end{bmatrix},$$

where the elements d_{ij} of $D_{r\times n}$ satisfy the following conditions:

(1) $d_{ij} = 0$ when $i > j$.

(2) The first non-zero entry in each row of $D_{r\times n}$ is 1.

(3) If $d_{ij} = 1$ is the first non-zero entry of the ith row then the jth column of D is the unit vector e_i whose only non-zero entry is in the ith position.

Definition 2.3 (The full rank factorization)

A matrix $A = BC \in \mathbb{C}^{m\times n}$ with $rank(A) = r$, is said to be a full rank factorization if B and C^T have r columns.

Definition 2.4 (The full column rank)

We say $A \in \mathbb{C}^{m\times n}$ has full column rank if the column rank of A is equal to the number of its columns, where the column rank is the maximum number of linearly independent columns.

Definition 2.5 (The full row rank)

We say $A \in \mathbb{C}^{m\times n}$ has full row rank if the row rank of A is equal to the number of its rows, where the row rank is the maximum number of linearly independent rows.

Definition 2.6 (Full rank)

An m by n matrix has full rank if $rank(A)$ equals the smaller of m and n.

Algorithm 2.1 To obtain the full rank factorization and MPGI for $A \in \mathbb{C}^{m\times n}$:

(1) Reduce A to row echelon form E_A.
(2) Select the distinguished columns of A (they are the columns that correspond to the columns $e_1, e_2, \dots, e_r$ in E_A) and place them as the columns in a matrix B in the same order as they appear in A.
(3) Select the non-zero rows from E_A and place them as rows in a matrix C in the same order as they appear in E_A.
(3) Compute $(CC^*)^{-1}$ and $(B^*B)^{-1}$.
(4) Compute $A^\dagger$ as $A^\dagger = C^*(CC^*)^{-1}(B^*B)^{-1}B^*$.

Theorem 2.1 If $A = BC$ where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$, and $r = rank(A) = rank(B) = rank(C)$, then

$$A^\dagger = C^*(CC^*)^{-1}(B^*B)^{-1}B^*.$$

3. AN ANALOGUES OF THE ADJUGATE MATRIX FOR MPGI

Determinant representations of MPGI for matrices that are based on analogues of the adjugate matrix were studied in [6-8, 10, 15]. We shall use some known facts [4, 8], and give some new lemmas and theorems.

Lemma 3.1 [6] For any matrix $A \in \mathbb{C}^{m\times n}$, there exists a unique Moore-Penrose inverse $A^\dagger$.

Theorem 3.1 A rectangular matrix has a right MPGI $A_+^\dagger$ if and only if it has full row rank.

Theorem 3.2 A rectangular matrix has a left MPGI $A_-^\dagger$ if and only if it has full column rank.

Corollary 3.1 If a rectangular matrix has either a left or a right MPGI, then A has MPGI $A^\dagger$.

Remarks 3.1

1) If $A^{\dagger}A \simeq I$, then $AA^{\dagger} \not\simeq I$.
2) If $AA^{\dagger} \simeq I$, then $A^{\dagger}A \not\simeq I$.

Lemma 3.2 Let $A \in \mathbb{C}^{m\times n}$, then

1. If A has full column rank, or full column rank of A equals the full rank, then $MPGI(A)$ has the following representation

$$A^{\dagger} = \frac{1}{|A^*A|}\begin{pmatrix} \sum a_{1j}A_{i1} & \sum a_{2j}A_{i1} & \cdots & \sum a_{mj}A_{i1} \\ \sum a_{1j}A_{i2} & \sum a_{2j}A_{i2} & \cdots & \sum a_{mj}A_{i2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{1j}A_{in} & \sum a_{2j}A_{in} & \cdots & \sum a_{mj}A_{in} \end{pmatrix}, i = \overline{1,m},\ j = \overline{1,n} \tag{3.1}$$

where A_{ij} is the cofactor of A^*A, $\sum a_{mj}A_{i1}$ is the determinant of the matrix A^*A after replacing its 1st column with the mth column of A^*. That is

$$\sum a_{mj}A_{i1} = |(A^*A)_{.1}a^*_{.m}|,$$

and $\sum a_{mj}A_{in}$ is the determinant of the matrix A^*A after replacing its nth column with the mth column of A^*. That is

$$\sum a_{mj}A_{in} = |(A^*A)_{.n}a^*_{.m}|.$$

So, we can simplify (3.1) into the form

$$A^{\dagger} = \frac{1}{|A^*A|}\begin{pmatrix} |(A^*A)_{.1}a^*_{.1}| & |(A^*A)_{.1}a^*_{.2}| & \cdots & |(A^*A)_{.1}a^*_{.m}| \\ |(A^*A)_{.2}a^*_{.1}| & |(A^*A)_{.2}a^*_{.2}| & \cdots & |(A^*A)_{.2}a^*_{.m}| \\ \cdots & \cdots & \cdots & \cdots \\ |(A^*A)_{.n}a^*_{.1}| & |(A^*A)_{.n}a^*_{.2}| & \cdots & |(A^*A)_{.n}a^*_{.m}| \end{pmatrix}. \tag{3.2}$$

2. If A has full row rank, or the full row rank of A equals the full rank, then $MPGI(A)$ has the following representation

$$A^{\dagger} = \frac{1}{|AA^*|}\begin{pmatrix} \sum a_{i1}A_{1j} & \sum a_{i1}A_{2j} & \cdots & \sum a_{i1}A_{mj} \\ \sum a_{i2}A_{1j} & \sum a_{i2}A_{2j} & \cdots & \sum a_{i2}A_{mj} \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{in}A_{1j} & \sum a_{in}A_{2j} & \cdots & \sum a_{in}A_{mj} \end{pmatrix}, i = \overline{1,m},\ j = \overline{1,n} \tag{3.3}$$

which is equivalent to

$$A^{\dagger} = \frac{1}{|AA^*|}\begin{pmatrix} |(AA^*)_{1.}a^*_{1.}| & |(AA^*)_{2.}a^*_{1.}| & \cdots & |(AA^*)_{m.}a^*_{1.}| \\ |(AA^*)_{1.}a^*_{2.}| & |(AA^*)_{2.}a^*_{2.}| & \cdots & |(AA^*)_{m.}a^*_{2.}| \\ \cdots & \cdots & \cdots & \cdots \\ |(AA^*)_{1.}a^*_{n.}| & |(AA^*)_{2.}a^*_{n.}| & \cdots & |(AA^*)_{m.}a^*_{n.}| \end{pmatrix}. \tag{3.4}$$

3. If A is a square matrix and has full rank, then $A^{\dagger} = A^{-1}$.

Lemma 3.2 If $A = BC \in \mathbb{C}^{m\times n}$ with $rank(A) = rank(B) = rank(C) = r$, such that A is a full rank factorization, then

$$A^{\dagger} = \frac{1}{|CC^*||B^*B|}\begin{pmatrix} \sum_{i=1}^{r}|(CC^*)_{i.}c_{1.}^*||(B^*B)_{.i}b_{.1}'^*| & \cdots & \sum_{i=1}^{r}|(CC^*)_{i.}c_{1.}^*||(B^*B)_{.i}b_{.m}'^*| \\ \cdots & \cdots & \cdots \\ \sum_{i=1}^{r}|(CC^*)_{i.}c_{n.}^*||(B^*B)_{.i}b_{.1}'^*| & \cdots & \sum_{i=1}^{r}|(CC^*)_{i.}c_{n.}^*||(B^*B)_{.i}b_{.m}'^*| \end{pmatrix}, \quad (3.5)$$

where $B = (b_{ij}') \in \mathbb{C}^{m\times r}, C = (c_{ij}) \in \mathbb{C}^{r\times n}$.

4. GENERALIZED CRAMER RULES USING MPGI

We know that Cramer's rule (1.2) gives an exact solution for non-singular square systems of linear equations [1]. In this section we give generalized Cramer rules using MPGI for solving the singular systems of linear equations (1.1). All solutions we obtained are least squares solutions with the minimum norm [2,4]. We give new proofs of generalized Cramer rules because it is very important in linear algebra.

Theorem 4.1 [2] Suppose that $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then $A^{\dagger}b$ is the minimal least squares solution to $Ax = b$.

Theorem 4.2 Let $A \in \mathbb{C}^{m\times n}$ in (1.1), then

(1) If A has full column rank, or the full rank equals full column rank, then the least squares solution with the minimum norm for the system (1.1) is

$$x_j = \frac{1}{|A^*A|}\sum_{k=1}^{m}|(A^*A)_{.j}\, a_{.k}^*|b_k, \quad j = \overline{1,n} \quad (4.1)$$

which is equivalent to the form

$$x_j = \frac{|(A^*A)_{.j}(A^*b)|}{|A^*A|}, j = \overline{1,n}\,. \quad (4.2)$$

(2) If A has full row rank, or the full rank equals full row rank, then the least squares solution with the minimum norm for the system (1.1) is

$$x_j = \frac{1}{|AA^*|}\sum_{k=1}^{m}|(AA^*)_{k.}a_{j.}^*|b_k \quad j = \overline{1,n}. \quad (4.3)$$

We have two proofs for this theorem.

Proof 1. If $A \in \mathbb{C}^{m\times n}$ has full column rank, or the full rank equals full column rank, then we use (3.1) or (3.2) (they are equivalent) and apply Theorem 4.1 to get the form (4.1) which is equivalent to (4.2).

Proof 2. If $A \in \mathbb{C}^{m\times n}$ has full column rank, or the full rank equals full column rank, then in this case use (1.4) and apply Theorem 4.1 as following

$$x = A^{\dagger}b$$

$$= (A^*A)^{-1}A^*b$$

$$= \frac{1}{|A^*A|}adj(A^*A)(A^*b)$$

So,

$$x_j = \frac{|(A^*A)_{.j}(A^*b)|}{|A^*A|}, j = \overline{1,n}\,.$$

The two poofs of (2) is analogous to the last proofs 1 and 2, but we will use (3.3) or (3.4) in proof 1 then apply Theorem 4.1. For the second proof, we use (1.6) (which satisfies (1.5)) as in proof 2, and do the same steps.

Theorem 4.3 If $A = BC \in \mathbb{C}^{m\times n}$ with $rank(A) = rank(B) = rank(C) = r$, such that A is a full rank factorization, then the least squares solution with the minimum norm for (1.1) is

$$x_j = \frac{\sum_{i=1}^{r}\left|(CC^*)_{i.}c_{j.}^*\right|\left|(B^*B)_{.i}b'^*_{.1}\right|b_1 + \sum_{i=1}^{r}\left|(CC^*)_{i.}c_{j.}^*\right|\left|(B^*B)_{.i}b'^*_{.2}\right|b_2 + \cdots + \sum_{i=1}^{r}\left|(CC^*)_{i.}c_{j.}^*\right|\left|(B^*B)_{.i}b'^*_{.m}\right|b_m}{|CC^*||B^*B|}. \quad (4.4)$$

Proof. Using Theorem 4.1 where $A^\dagger$ is given by (3.5), then we get (4.4).

Note that, we can apply Theorem 4.3 when $rank(A) = r < \min\{m, n\}$.

5. NUMERICAL EXAMPLES

In this section, we give some examples to illustrate our main results.

Example 5.1 Consider the following system

$$x_1 + 3x_2 = 17,$$

$$5x_1 + 7x_2 = 19,$$

$$11x_1 + 13x_2 = 23.$$

We can write this system in the form (1.1), where

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix} \in \mathbb{C}^{3\times 2}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, b = \begin{pmatrix} 17 \\ 19 \\ 23 \end{pmatrix}.$$

Since $rank(A) = 2 = n(< m = 3)$, then A has full column rank (also, it has full rank), in this case, we use the form (4.1) or (4.2).

$$A^* = \begin{pmatrix} 1 & 5 & 11 \\ 3 & 7 & 13 \end{pmatrix}, \qquad A^*A = \begin{pmatrix} 147 & 181 \\ 181 & 227 \end{pmatrix}, \qquad |A^*A| = 608.$$

If we use (4.1) then

$$x_1 = \frac{|(A^*A)_{.1}a_{.1}^*|b_1 + |(A^*A)_{.1}a_{.2}^*|b_2 + |(A^*A)_{.1}a_{.3}^*|b_3}{|A^*A|},$$

$$x_2 = \frac{|(A^*A)_{.2}a_{.1}^*|b_1 + |(A^*A)_{.2}a_{.2}^*|b_2 + |(A^*A)_{.2}a_{.3}^*|b_3}{|A^*A|};$$

where

$$|(A^*A)_{.1}a_{.1}^*|b_1 = \begin{vmatrix} 1 & 181 \\ 3 & 227 \end{vmatrix}(17) = -5372,$$

$$|(A^*A)_{.1}a_{.2}^*|b_2 = \begin{vmatrix} 5 & 181 \\ 7 & 227 \end{vmatrix}(19) = -2508,$$

$$|(A^*A)_{.1}a_{.3}^*|b_3 = \begin{vmatrix} 11 & 181 \\ 13 & 227 \end{vmatrix}(23) = 3312,$$

$$|(A^*A)_{.2}a^*_{.1}|b_1 = \begin{vmatrix}147 & 1\\ 181 & 3\end{vmatrix}(17) = 4420,$$

$$|(A^*A)_{.2}a^*_{.2}|b_2 = \begin{vmatrix}147 & 5\\ 181 & 7\end{vmatrix}(19) = 2356$$

$$|(A^*A)_{.2}a^*_{.3}|b_3 = \begin{vmatrix}147 & 11\\ 181 & 13\end{vmatrix}(23) = -1840.$$

Now,

$$x_1 = \frac{-5372 - 2508 + 3312}{608} = -7.5,$$

$$x_2 = \frac{4420 + 2356 - 1840}{608} = 8.12.$$

Remark 5.1 About example 5.1**,** we obtained the approximate solution. Furthermore, you can use the form (4.2) and will get the same results. The solution we obtained is the same solution obtained using the method used in [4].

Example 5.2 Let us consider the system

$$2x_1 + 3x_2 - 4x_3 = 7,$$

$$x_1 - 2x_2 - 5x_3 = 3.$$

We can write this system in the form (1.1), where

$$A = \begin{pmatrix}2 & 3 & -4\\ 1 & -2 & -5\end{pmatrix},\quad x = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}, b = \begin{pmatrix}7\\ 3\end{pmatrix}.$$

Since $rank(A) = 2 = m(< n = 3)$, then A has full row rank (also, it has full rank), in this case, we use the form (4.3) to find x as following.

$$A^* = \begin{pmatrix}2 & 1\\ 3 & -2\\ -4 & -5\end{pmatrix},\qquad AA^* = \begin{pmatrix}29 & 16\\ 16 & 30\end{pmatrix},\qquad |AA^*| = 614.$$

$$x_1 = \frac{|(AA^*)_{1.}a^*_{1.}|b_1 + |(AA^*)_{2.}a^*_{1.}|b_2}{|AA^*|},$$

$$x_2 = \frac{|(AA^*)_{1.}a^*_{2.}|b_1 + |(AA^*)_{2.}a^*_{2.}|b_2}{|AA^*|},$$

$$x_3 = \frac{|(AA^*)_{1.}a^*_{3.}|b_1 + |(AA^*)_{2.}a^*_{3.}|b_2}{|AA^*|};$$

where

$$|(AA^*)_{1.}a^*_{1.}|b_1 = \begin{vmatrix}2 & 1\\ 16 & 30\end{vmatrix}(7) = 308,$$

$$|(AA^*)_{2.}a^*_{1.}|b_2 = \begin{vmatrix}29 & 16\\ 2 & 1\end{vmatrix}(3) = -9,$$

$$|(AA^*)_{1.}a^*_{2.}|b_1 = \begin{vmatrix}3 & -2\\ 16 & 30\end{vmatrix}(7) = 854.$$

$$|(AA^*)_{2.}\, a^*_{2.}|b_2 = \begin{vmatrix} 29 & 16 \\ 3 & -2 \end{vmatrix}(3) = -318,$$

$$|(AA^*)_{1.}\, a^*_{3.}|b_1 = \begin{vmatrix} -4 & -5 \\ 16 & 30 \end{vmatrix}(7) = -280,$$

$$|(AA^*)_{2.}\, a^*_{3.}|b_2 = \begin{vmatrix} 29 & 16 \\ -4 & -5 \end{vmatrix}(3) = -243.$$

Hence

$$x_1 = 0.49,\ x_2 = 0.87,\ x_3 = -0.85,$$

is the solution for the given system.

Example 5.3 Let us consider the system

$$x_1 + 2x_2 + 4x_3 = 2,$$

$$2x_1 + 4x_2 + 8x_3 = 4.$$

The coefficients matrix is

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}.$$

Since $rank(A) = 1 < \min\{m, n\}$, then A has not full rank. It is a full rank factorization. We can write A as

$$A = BC$$

with $rank(A) = rank(B) = rank(C) = 1$, where

$$B = \begin{pmatrix} b'_{11} \\ b'_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}.$$

To find solution for the given system, we use the form (4.4), where

$$x_1 = \frac{|(CC^*)_{1.}\, c^*_{1.}|\,|(B^*B)_{.1}\, b'^{*}_{.1}|\,b_1 + |(CC^*)_{1.}\, c^*_{1.}|\,|(B^*B)_{.1}\, b'^{*}_{.2}|\,b_2}{|CC^*||B^*B|},$$

$$x_2 = \frac{|(CC^*)_{1.}\, c^*_{2.}|\,|(B^*B)_{.1}\, b'^{*}_{.1}|\,b_1 + |(CC^*)_{1.}\, c^*_{2.}|\,|(B^*B)_{.1}\, b'^{*}_{.2}|\,b_2}{|CC^*||B^*B|},$$

$$x_3 = \frac{|(CC^*)_{1.}\, c^*_{3.}|\,|(B^*B)_{.1}\, b'^{*}_{.1}|\,b_1 + |(CC^*)_{1.}\, c^*_{3.}|\,|(B^*B)_{.1}\, b'^{*}_{.2}|\,b_2}{|CC^*||B^*B|},$$

$$B^* = \begin{pmatrix} 1 & 2 \end{pmatrix},\ B^*B = (5),\ |B^*B| = 5,\ (B^*B)^{-1} = \frac{1}{5},$$

$$C^* = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, CC^* = (21), |CC^*| = 21,\ (CC^*)^{-1} = \frac{1}{21}.$$

Hence

$$x_1 = \frac{(1)(1)(2) + (1)(2)(4)}{(21)(5)} = \frac{10}{105},$$

$$x_2 = \frac{(2)(1)(2)+(2)(2)(4)}{(21)(5)} = \frac{20}{105},$$

$$x_3 = \frac{(4)(1)(2)+(4)(2)(4)}{(21)(5)} = \frac{40}{105}.$$

6.CONCLUSIONS

In this study, we obtained the representations for MPGI of some matrices using the adjugate matrix. Then we used these representations to get generalized Cramer rules. Finally, we used the generalized Cramer rules to find the minimal least squares solutions for the singular linear systems of algebraic equations (1.1).

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