Cleavability over Productive and hereditary topological spaces

Ghazeel .Almahdi. Jalalah[1 (*)] *, Hawa Alafitouri Mohammed Alwar.*[2]
[1]*dept. of Mathematics, Faculty of Education , Sirte University, Libya.*
[2] *Faculty of Engineering. Zawia University .Libya.*[2]

Abstract

In this paper new kind of cleavability was introduced over some topological spaces called μ-cleavability and studied some basic properties of concept of cleavability of a space over productive and hereditary class to show when it belongs to this class.

Key words:*μ_pointwise cleavable space, E-cleavable spaces, μ-double cleavable space, absolutely cleavable space.*

(∗) Email: gjellala@yahoo.com

1. Introduction

Different types of cleavability (originally named "splittability ") of Atopological space were introduced by Arhangl' Skii (1985) as following: Atopological space X is said to be cleavable (or splitable) over a class of spaces $\mathcal{P}$ if for $A \subset X$ there exists a continuous mapping:

$f: X \to Y \in \mathcal{P}$ such that $f^{-1}f(A) = A,\ f(X) = Y$.

Throughout this paper (X, τ), (Y, σ) (or simply X, and Y) denote topological spaces which no separation axioms are assumed otherwise mentioned.

2-Preliminaries

In this section, we recall some definitions and theorems which we needed in this paper.

Definition(3-1) [5]

Let $P_x: \mathbf{X} \times \boldsymbol{Y} \longrightarrow \mathbf{X}$ be defined by the equation $P_x(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$.

The maps P_x and P_y are called projections of $\boldsymbol{X} \times \boldsymbol{Y}$and

$P_y: \boldsymbol{X} \times \boldsymbol{Y} \longrightarrow \boldsymbol{Y}$ be defined by P_y $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$, onto its first and second $\boldsymbol{P_y}$ factors respectively.

Theorem (3-2) [6]

Let $\boldsymbol{f}$:X$\longrightarrow$ $\mathbf{Y}$ be continuous mapping between two topological

spaces and $\mathbf{X} \times \mathbf{Y}$ be a product space, and the graph $\boldsymbol{G} =\{(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{Y},\ \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\}$ then $\boldsymbol{G}$ is homeomorphic to $\boldsymbol{X}$,

If A$\subset \boldsymbol{X}$ and $\boldsymbol{g}$:A$\longrightarrow$ X$\times \boldsymbol{g}$(A) then $\boldsymbol{g}$ is well defined and continuous.

Lemma (3-3)

The projection $\boldsymbol{P}_x$, $\boldsymbol{P}_y$ from the product space $\boldsymbol{X} \times \boldsymbol{Y}$ onto $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively are continuous and open .

Definition (3-4) [4]

Atopological property is called hereditary if it carries over from a space to its subspaces, i-e if a space $\boldsymbol{X}$ has this property, then each subspace of $\boldsymbol{X}$ has it. As T_0, T_1, T_2 (spaces) and regular space are hereditary.

Definition (3-5) [5]

Let $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be an injective continuous map, where $\boldsymbol{X}$ and $\boldsymbol{Y}$are topological spaces. Let Z be image set of $\boldsymbol{f}(\boldsymbol{x})$. Considered a subspace of Y, then function $\boldsymbol{f}':\boldsymbol{X} \longrightarrow \boldsymbol{Z}$ obtained by restricting the range of f to be bijective. If $\boldsymbol{f}'$ happens to be homeomorphism of X with Z, we say that the map $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is a topological imbedding, or simply an imbedding of X in Y.

Definition(3-6)[1]

Atopological space X is said to be pointwise Cleavable over a class of spaces $\boldsymbol{\mathcal{P}}$ if for any $x \in X$ there exists a continuous mapping $f: X \rightarrow Y$ such that $f^{-1}f(x) = x$.

Definition(3-7)[3]

Atopological space $\boldsymbol{X}$ is said to be absolutely cleavable over a

class of spaces $\boldsymbol{\mathcal{P}}$, if $\boldsymbol{A} \subset \boldsymbol{X}$, and there exists an injective continuous mapping $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y} \in \boldsymbol{\mathcal{P}}$, such that $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{A}) = \boldsymbol{A}$.

Definition(3-8)[2]

Atopological space $\boldsymbol{X}$ is said to be double cleavable over a class of spaces $\boldsymbol{\mathcal{P}}$, if for any $\boldsymbol{A} \subset \boldsymbol{X}$, and $\boldsymbol{B} \subset \boldsymbol{X}$ there exists a continuous mapping $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y} \in \boldsymbol{\mathcal{P}}$, such that $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{A}) = \boldsymbol{A}$ and $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{B}) = \boldsymbol{B}$.

3 - µ-cleavability and E-cleavability

To say topological space X is cleavable over µ-class of topological spaces, if for every subset A of X there is a space $\mu_A \in \mu$ and a continuous function $f_A: X \to \mu_A$, such that if $x \in A$ and $y \in X/A$ then $f_A(X) \neq f_A(y)$, then the function f_A is called a cleaving function for A.

Definition(4-1)

Aspace $\boldsymbol{X}$ is called $\boldsymbol{\mu}$-cleavable over $\boldsymbol{P}$ If for every subset $A \subset X$ there exists a map $f_A:\boldsymbol{X} \longrightarrow \boldsymbol{Y}_A$ such that $f_A(\boldsymbol{X}) = \boldsymbol{Y} \in \boldsymbol{P}$ and

$$f_A^{-1} f_A(\boldsymbol{A}) = \boldsymbol{A}.$$

where $\boldsymbol{P}$ is the class of all spaces of Y and $\boldsymbol{\mu}$ is the class of all continuous mappings.

Definition(4-2)

Atopological space $\boldsymbol{X}$ is said to be pointwise $\boldsymbol{\mu}$-cleavable over $\boldsymbol{P}$ if for every point $\boldsymbol{x} \in \boldsymbol{X}$, there exists $\boldsymbol{f} \in \boldsymbol{\mu}$, $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}$, where $\boldsymbol{Y} \in \boldsymbol{P}$ such That $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.

Definition(4-3)

Atopological space X is said to be $\boldsymbol{\mu}$-double cleavable over $\boldsymbol{P}$ if for every pair subsets $\boldsymbol{A}$ and $\boldsymbol{B}$ of $\boldsymbol{X}$ there exists $\boldsymbol{f} \in \boldsymbol{\mu}$, $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}$, where $\boldsymbol{Y} \in \boldsymbol{P}$ such that $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{A}) = \boldsymbol{A}$ and $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{B}) = \boldsymbol{B}$.

Definition(4-4)

Atopological space $\boldsymbol{X}$ is said to be absolutely cleavable over class of spaces $\boldsymbol{\mathcal{P}}$, if for any subset $\boldsymbol{A}$ of $\boldsymbol{X}$, there exists an injective continuous mapping $\boldsymbol{f} \in \boldsymbol{\mu}$, $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}$, such that $\boldsymbol{f}^{-1}\boldsymbol{f}(\boldsymbol{A}) = \boldsymbol{A}$.

Note that if $\boldsymbol{\mathcal{P}}$ is the class of all spaces, we shall say that $\boldsymbol{X}$ is absolutely cleavable over $\boldsymbol{\mathcal{P}}$. If $\boldsymbol{f} \in \boldsymbol{\mu}$ is an open, closed, perfect,…(continuous)

mapping, we shall say that $\boldsymbol{X}$ respectively open, closed perfectabsolutely cleavable over $\boldsymbol{\mathcal{P}}$.

Note that if $\boldsymbol{f}$ is an injective continuous map of $\boldsymbol{X}$ into $\boldsymbol{Y} \in \boldsymbol{\mathcal{P}}$, then $\boldsymbol{X}$is cleavable over $\boldsymbol{\mathcal{P}}$, and since the definition of cleavability depends on the subset $\boldsymbol{A}$ of $\boldsymbol{X}$, thus we might say a space $\boldsymbol{X}$ is said to be absolutely cleavable over $\boldsymbol{\mathcal{P}}$, then the cleavability over $\boldsymbol{\mathcal{P}}$, may regarded as generalization of continuous injection map onto $\boldsymbol{Y} \in \boldsymbol{\mathcal{P}}$.

Remark(4-5)

In analogy with $\boldsymbol{\mu}$-cleavability even for $\boldsymbol{\mu}$-pointwise and $\boldsymbol{\mu}$–double cleavability we mean that pointwise (double) cleavability or open closed pointwise (double) cleavability over $\boldsymbol{P}$.

Proposition(4-6)

Let $\wp$ be a productive class of spaces. the following conditions are equivalent:

(1) $\boldsymbol{X}$ is point wise cleavable over ; $\wp$

(2) $\boldsymbol{X}$ is cleavable over $\wp$;

(3) $\boldsymbol{X}$ is double cleavable over $\wp$;

(4) $\boldsymbol{X}$ is absolutely cleavable over $\wp$.

Proof:

It sufficient to prove that (1) $\longrightarrow$ (4)

Let $\boldsymbol{x} \in \boldsymbol{X}$ then there exists a space $\boldsymbol{Y}_x \in \wp$ and a continuous mapping

$f_x : \boldsymbol{X} \longrightarrow \boldsymbol{Y}_x$, such that $f_x^{-1} f_x\{\boldsymbol{x}\} = \{\boldsymbol{x}\}$. To a point $\mathbf{z} \in \mathbf{X}$ we assign the

$\boldsymbol{g}(\boldsymbol{Z}) = \{\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{Z})\}_{\boldsymbol{x}\in\boldsymbol{X}} \in \prod_{x\in X} \boldsymbol{Y}_x \in \wp$,then the mapping $\boldsymbol{g}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}_x$ defined

In this way is one-to-one and continuous.

Remark (4-7)

In the following when the class $\wp$ of space is productive. we use the term E-cleavable over $\wp$ to indicate one of the four equivalent forms of cleavability of the proposition (4-6)

Definition (4-8) [5]

A class $\boldsymbol{P}$ of topological spaces is said to be expansive if for every $\boldsymbol{X} \in \boldsymbol{P}$ and continuous bijection $\boldsymbol{f}:\boldsymbol{Y} \longrightarrow \boldsymbol{X}$ then $\boldsymbol{Y} \in \boldsymbol{P}$.

Corollary(4-9)

Let $\wp$ be a productive class of spaces, if $\wp$ is also hereditary and $\boldsymbol{X}$ is E-cleavable over $\wp$, then $\boldsymbol{X} \in \wp$.

Proof:

By the proposition (4-6) there exists a one-to-one continuous Mapping $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{Y}$ where $\boldsymbol{Y} \in \wp$ since $\wp$ is hereditary. it follows that

$\boldsymbol{f}(\boldsymbol{X}) \in \wp$. So $\boldsymbol{f}:\boldsymbol{X} \longrightarrow \boldsymbol{f}(\boldsymbol{X})$ is continuous bijection and then by definition (4-8) then $\boldsymbol{X} \in \wp$.

Property(4-10) [7]

Let $\wp$ be a productive class of spaces, If $\boldsymbol{X}$ is pointwise closed cleavable over $\wp$, then $\boldsymbol{X}$ can be embedded as subspace of some space of $\wp$.

Proof:

By hypothesis every continuous mapping $\boldsymbol{f}_x:\boldsymbol{X} \longrightarrow \boldsymbol{Y}_x$ such that

$\{\boldsymbol{x}\} = f_x^{-1} f \{\boldsymbol{x}\}$ is closed and then $\boldsymbol{g}:\boldsymbol{X} \longrightarrow \boldsymbol{g}(\boldsymbol{X})$ is closed mapping, So, we have to prove this fact;

Let $\boldsymbol{A} \subset \boldsymbol{X}$ be closed we want to prove that $\boldsymbol{g}(\boldsymbol{A}) = \prod_{x\in X} \boldsymbol{f}_x(\boldsymbol{A}) \bigcap \boldsymbol{g}(\boldsymbol{X})$ where $\boldsymbol{f}_x(\boldsymbol{A})$ is closed subset of $\prod_{x\in X} \boldsymbol{Y}_x$.

The inclusion $\boldsymbol{g}(\boldsymbol{A}) \subset \prod_{x\in X} \boldsymbol{f}_x(\boldsymbol{A}) \bigcap \boldsymbol{g}(\boldsymbol{X})$ is obvious.

Now, let $\boldsymbol{y} \in \prod_{x\in X} \boldsymbol{f}_x(\boldsymbol{A}) \bigcap \boldsymbol{g}(\boldsymbol{X})$ and $\boldsymbol{z} \in \boldsymbol{X}$ such that $\boldsymbol{g}(\boldsymbol{z}) = \boldsymbol{y}$,

then $\boldsymbol{g}(\boldsymbol{z}) = \{\boldsymbol{f}_x(z)\}_{x\in X} \in \prod_{x\in X} \boldsymbol{f}_x(\boldsymbol{A})$. For every $\boldsymbol{x} \in \boldsymbol{X}$ we have to show that $\boldsymbol{f}_x(\boldsymbol{z}) \in \boldsymbol{f}_x(\boldsymbol{A})$, so $\boldsymbol{f}_z(\boldsymbol{z}) \in \boldsymbol{f}_z(\boldsymbol{A})$, then there exists $\boldsymbol{a} \in \boldsymbol{A}$ such that $\boldsymbol{f}_z(\boldsymbol{z})=\boldsymbol{f}_z(\boldsymbol{a})$ and by hypothesis ($\boldsymbol{f}$ is 1-1), $\boldsymbol{z} \in \boldsymbol{A}$ and so $\boldsymbol{y} \in \boldsymbol{g}(\boldsymbol{A})$.

The proof is complete.

Remark(4-11)

If ℘ is a productive and hereditary class of spaces, then the property (4-10) is equivalent to say that if ***X*** is pointwise closed cleavable over ℘ then ***X*** is absolutely closed cleavable over ℘.

4- Conclusion

We have the following two results:

1) Let ℘be a productive class of spaces, if ℘ is hereditary and ***X*** is closed E-cleavable over ℘ then ***X*** ∈ ℘.

2) We can apply the property (4-10) on some spaces as T_0 , T_1, T_2 spaces, but not on normal spaces because they are not productive spaces in general.

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