Incompressible Euler-stratified system with a function $\boldsymbol{F} \in C^1(\mathbb{R}, \mathbb{R}^2)$

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Abstract

In this paper, we prove the unique global weak solution for the Euler-stratified system with a vector valued function $F \in C^1(\mathbb{R}, \mathbb{R}^2)$ *satisfying* $F(0) = 0$, *when bounded vorticity in the* $LBMO$ *spaces (see Definition 2.5). In this category, we use the approach of* [8].

Keywords *: Incompressible fluid flow, global weak solution, Euler stratified system.*

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1. Introduction

Euler stratified system for the incompressible fluid flow in $\mathbb{R}^2$ with a function F is of the form :

$$(1.1)\qquad \begin{cases} \partial_t v + v.\nabla v + \nabla p = F(\theta), & (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0, & \\ div\, v = 0, \quad (v,\theta)|_{t=0} = (v_0, \theta_0), & \end{cases}$$

where $v = v(x,t)$, $x \in \mathbb{R}^2$, $t \in \mathbb{R}_+$ with $v = (v_1, v_2)$ is the velocity vector field. The differential operator $v.\nabla$ is defined by :

$$v.\nabla = \sum_{i=1}^{d} v_i \partial_i.$$

The scalar pressure $p = p(x,t)$ is defined by :

$$\Delta p = -div\,(v.\nabla v).$$

The function $F(\theta) = (F_1(\theta), F_2(\theta))$ is a vector valued function such that $F \in C^1(\mathbb{R}, \mathbb{R}^2)$ where $F(0) = 0$ and $\boldsymbol{\theta}$ is the scalar temperature .The second equation of (1.1) is called the Fourier equation models the phenomenon of conservation and dissipation. It is called also a transport-diffusion equation. The condition $div\, v = 0$ explains that the fluid is incompressible.

Note that if we take $F(\theta) = \theta e_2$, where e_2 is the vector defined by $e_2 = (0,1)$, then the system (1.1) coincide with the classical incompressible Euler stratified equations, that is :

$$(1.2)\qquad \begin{cases} \partial_t v + v.\nabla v + \nabla p = \theta e_2, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0, & \\ div\, v = 0, \quad (v,\theta)|_{t=0} = (v_0, \theta_0), & \end{cases}$$

The system (1.2) is studied by many authors in a different functional spaces, see [8], where the global weak solution for (1.2) in the LBMO spaces is proved.

If the initial temperature $\boldsymbol{\theta}_0$ is identically constant, we obtain the Euler system given by,

$$(1.3)\qquad \begin{cases} \partial_t v + v.\nabla v + \nabla p = 0, \\ div\, v = 0, \\ v|_{t=0} = v_0 \end{cases}$$

The question of local well-posedness of (1.3) with smooth data was resolved by many authors in different spaces see for instance [5,7]. In this context, the vorticity $\omega = curl\, v$ played a fundamental role. In fact, the well-known BKM criterion [4] ensures that the development of finite time singularities for these solutions is related by the blow-up of the L^∞ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of the two-dimensional Euler solutions with smooth initial data, since the vorticity satisfies the equation

$$\partial_t \omega + v.\nabla\omega = 0,$$

and then all the L^p norms are conserved. The global well posedness result for the system (1.3), is proved in a different spaces and we focus on [3], where the authors proved a similar result for system (1.3) in the $LBMO$ spaces.

We turn to the system (1.2) and note that the global well-posedness result for this system was solved by many authors and in a different functional spaces see for instance [9,10].

In this paper, we extend a global well-posedness or (a unique global weak solution) result for the system (1.1) with a bounded vorticity in the $LBMO$ spaces and we will use the same approach of [3,8]. Our main result is the following.

Theorem 1.1 Given $F \in C^1(\mathbb{R}, \mathbb{R}^2)$, satisfy $F(0) = 0$. Let v_0 be a divergence-free vector field of vorticity $\omega_0 \in L^2 \cap LBMO$. Let also

$\theta_0 \in L^2 \cap L^\infty$ a real-valued function. Then there exists a unique global weak solution (v, θ) for the system (1.1).

Moreover, there exists a constant C_0 depending only on the $L^2 \cap LBMO$ norm of ω_0 and the $L^2 \cap L^\infty$ norm of θ_0 such that

$$\|\omega(t)\|_{LBMO \cap L^2} \leq C_0 e^{C_0 t} \qquad (1.4)$$

Let us now give some remarks about the main idea of the proof of our theorem.

Remark 1.1 To establish a classical L^2 −estimate for ω equation, we shall need to estimate the composition $\grave{F}_i \circ \theta$ in L^∞ space. Indeed, the vorticity for the system (1.1) satisfies the equation

$$\begin{cases} \partial_t \omega + v.\nabla\omega = \partial_1(F_2(\theta)) - \partial_2(F_1(\theta)), \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0, \\ (\omega, \theta)_{t=0} = (\omega_0, \theta_0). \end{cases}$$

Taking the L^2 scalar product, we get successively :

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \sum_{i=1}^{2} \int_0^t \left\|\grave{F}_i \circ \theta(\tau)\right\|_{L^\infty} \|\nabla\theta(\tau)\|_{L^2} d\tau.$$

We use the fact θ which is transported by the flow and the following theorem which treats the action of composition law with smooth functions in the Besov spaces (see Definition 2.6). It plays a significant role in the sequel. The proof can be found in [1].

Theorem 1.2 Let $F \in C^{[s]+2}$ with $F(0) = 0$ and $s \in [0, \infty]$. Assume that $\theta \in B^s_{p,r} \cap L^\infty$ with $(p, r) \in [1, \infty]^2$, then $F \circ \theta \in B^s_{p,r}$ and satisfying :

$$\|F \circ \theta\|_{B^s_{p,r}} \leq C_s sup_{|x|<C\|\theta\|_{L^\infty}} \left\|F^{[s]+2}(x)\right\|_{L^\infty} \|\theta\|_{B^s_{p,r}}.$$

Since $F \in \mathrm{C}^1(\mathbb{R}, \mathbb{R}^2)$, and using this theorem, with the assumputation $\theta_0 \in L^\infty$, and the embeddings $B^0_{\infty,\infty} \hookrightarrow \dot{B}^0_{\infty,\infty} \hookrightarrow L^\infty$ we get :

$$\left\|\grave{F}_i \circ \theta\right\|_{L^\infty} \leq sup_{|x|\leq C\|\theta_0\|_{L^\infty}} \|\nabla F_i\|_{L^\infty} \leq C.$$

Therefore we obtain :

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\nabla F\|_{L^\infty} \int_0^t \|\nabla\theta(\tau)\|_{L^2} d\tau$$

$$\leq \|\omega_0\|_{L^2} + C \int_0^t \|\nabla\theta(\tau)\|_{L^2} d\tau$$

and

$$\|\theta(t)\|_{L^2} + \|\nabla\theta(t)\|_{L^2} = \|\theta_0\|_{L^2}.$$

We use to prove (1.4) a logarithmic estimate in the space $L^2 \cap LBMO$ see Theorem 2 in [3], that we recall it in section 3 (Theorem 3.2 in this paper).

Remark 1.2

From the proof, we can conclude that

$$\|v(t)\|_{LL} \leq C_0 e^{C_0\, t}, \quad \forall t \in \mathbb{R}_+,$$

where LL is the space of log-Lipschitz functions see (Definition 2.2).

The paper is organized as follows. In section 2, we give some definitions and recall some functional spaces. Section 3 is devoted to recall some properties of the $LBMO$ spaces and in section 4, we prove our main result (Theorem 1.1).

2. Technical Tools

In this section, we recall some notations and some functional spaces as a Lebesgue space L^p, the space LL of log-Lipschitz functions, the space BMO of bounded mean oscillations function and $LBMO$ spaces.

Also we give the definition of Besov space and some results used in the paper.

2.1 Notation

- We denote by C any positive constant than will change from line to line and C_0 a real positive constant depending on the size of the initial data.
- For any A and B, we say that $A \lesssim B$, if there exist a constant $C > 0$ such that $A \leq CB$.
- The space C_0^∞ is the space of all continuous function.
- For all set $D \subset \mathbb{R}^2$ and every integrable function f, we define $Avg_D(f)$ by the relation :

$$Avg_D(f) \coloneqq \frac{1}{|D|} \int_D f(x)dx.$$

2.2 Some functional spaces

This section is devoted to recall some functional spaces and gives the lemma of Gronwall. Also, a result about the flow of the velocity will be given.

Definition 2.1 we define the usual Lebesgue space $L^p(\mathbb{R}^d), p \in [1, +\infty[$, by the space of all function f such that :

$$\|f\|_{L^p} \coloneqq \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

and for $p = \infty$, we say that $f \in L^\infty$, if

$$\|f\|_{L^\infty} \coloneqq sup_x |f(x)| < \infty.$$

Definition 2.2 We define the space LL of log-Lipschitz functions by the space of the set of bounded vector fields f such that :

$$\|f\|_{LL} \coloneqq sup_{0<|x-y|\leq 1} \frac{|f(x)-f(y)|}{|x-y|(1+|\log(|x-y|)|)} < \infty.$$

Definition 2.3 For every homeomorphism ψ, we set :

$$\|\psi\|_{*} := sup_{x\neq y} \Phi(|\psi(x)-\psi(y)|, |x-y|),$$

where Φ is defined on $]0,\infty[\times]0,\infty[$ by :

$$\Phi(r,s) = \begin{cases} \max\left\{\frac{1+|\ln s|}{1+|\ln r|}, \frac{1+|\ln r|}{1+|\ln s|}\right\}, & if\ (1-s)(1-r)\geq 0 \\ (1+|\ln s|)(1+|\ln r|)\ , & if\ \ (1-s)(1-r)\leq 0. \end{cases}$$

Since Φ is symmetric, then $\|\Phi\|_{*} = \|\psi^{-1}\|_{*} \geq 1$. It is clear that every homeomorphism Φ satisfies :

$$\frac{1}{C}|x-y|^{\alpha} \leq |\psi(x)-\psi(y)| \leq C|x-y|^{\beta},$$

for some $\alpha, \beta, C > 0$ has its $\|\psi\|_{*}$ finite.

Definition 2.4 The space $BMO(\mathbb{R}^d)$ of bounded mean oscillations is the set of locally integrable functions f such that :

$$\|f\|_{BMO} := sup_{B} Avg_{B}|f - Avg_{B}(f)| < \infty,$$

where B is a ball in $\mathbb{R}^2$ with $\sup B > 0$.

Definition 2.5 We say that $f \in LBMO$ if and only if

$$\|f\|_{LBMO} \coloneqq \|f\|_{BMO} + sup_{B_1,B_2} \frac{\left|Avg_{B_2}(f) - Avg_{B_1}(f)\right|}{1+\ln\left(\frac{1-\ln r_{B_2}}{1-\ln r_{B_1}}\right)} < \infty,$$

where B_1 and B_2 are balls in $\mathbb{R}^2$ with $0 < r_{B_1} \leq 1$ and $2B_2 \subset B_1$.

We need the definition of Besov space. We define the dyadic decomposition of the full space $\mathbb{R}^2$ and recall the Littlewood-Paley operators, see for example [5]. There exist two nonnegative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2/\{0\})$ such that :

$$\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$$

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \forall \xi \in \mathbb{R}^2/\{0\},$$

$$|p-q| \geq 2 \Rightarrow supp\ \varphi(2^{-p}.) \cap upp\ \varphi(2^{-q}.) = \phi,$$

$$q \geq 1 \Rightarrow supp\ \chi \cap upp\ \varphi(2^{-q}.) = \phi.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\bar{h} = \mathcal{F}^{-1}\chi$, the frequency localization operators Δ_q and S_q are defined by :

$$\Delta_q f = \varphi(2^{-q}D)f, \qquad S_q f = \chi(2^{-q}D)f$$

$$\Delta_{-1} f = S_0 f, \qquad \Delta_q f = 0 \quad for\ q \leq -2.$$

The homogeneous operators are defined as follows

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q f = \varphi(2^{-q}D)f,$$

We recall now the definition of Besov spaces, see [3,5].

Definition 2.6 (Besov space)

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. The inhomogeneous Besov space $B^s_{p,r}$ defined by :

$$B^s_{p,r} = \left\{ f \in \acute{S}(\mathbb{R}^2) : \|f\|_{B^s_{p,r}} < \infty \right\},$$

where S is the Schwartz space and

$$\|f\|_{B^s_{p,r}} := \left(2^{qs}\left\|\Delta_q f\right\|_{L^p}\right)_{l^r}.$$

The homogeneous norm :

$$\|f\|_{B^s_{p,r}} := \left(2^{qs}\left\|\dot{\Delta}_q f\right\|_{L^p}\right)_{l^r}.$$

The following proposition was proved in [3].

Proposition 2.1 Let v be a smooth divergence-free vector field and ψ its flow,

$$\begin{cases} \partial_t \psi(t,x) = v(t,\psi(t,x)) \\ \psi(0,x) = x. \end{cases}$$

Then for every $t \geq 0$ we have :

$$\|\psi(t,.)\|_* \leq \exp\left(\int_0^t \|v(\tau)\|_{LL} d\tau\right).$$

The following lemma is needed in the proof of our main result see [2] for a proof.

Lemma 2.1 (Gronwall's lemma)

Let f is a nonnegative continuous function on $[0,t]$, a is a real number and let A be a continuous function on $[0,t]$. Suppose also that :

$$f(t) \leq a + \int_0^t A(\tau) f(\tau) d\tau.$$

Then we have :

$$f(t) \leq a\, exp\left(\int_0^t A(\tau) d\tau\right).$$

3. Some results on the $LBMO$ space

We state some properties of the $LBMO$ spaces. We introduce the following proposition proved in [3].

Proposition 3.1 The following properties holds :

(1) The space $LBMO$ is a Banach space included in BMO and strictly containing $L^\infty(\mathbb{R}^2)$.

(2) For every $g \in C_0^\infty(\mathbb{R}^2)$ and $f \in LBMO$, we have :

$$\|g * f\|_{LBMO} \leq \|g\|_{L^1} \|f\|_{LBMO}.$$

The following theorem is the main ingredient for proving Theorem 1.1, see [3].

Theorem 3.2 There exists a universal constant $C > 0$ such that,

$$\|f \circ \psi\|_{LBMO \cap L^2} \leq C \, \|f\|_{LBMO \cap L^2} \ln(1 + \|\psi\|_*)$$

for any Lebesgue measure preserving homeomorphism ψ.

4. Proof of main result

This section is devoted to the proof of Theorem 1.1 which can be divided in three steps.

- **First step :** A priori estimates which are the main ingredients for the proof of our main result.
- **Second step :** The existence of the solutions (v, θ) of the system (1.1).
- **Third step :** The uniqueness of the solutions (v, θ) of the system (1.1).

We only discuss the first step, that is, we only prove some a priori estimates which are the main ingredients for the proof of our main result. The second and the third are standard, see [2], [3], [6], [7], [11] and [12].

Proposition 4.1

Let (v, θ) be a smooth solution of the system (1.1) with vorticity ω. Assume that $F \in C^1(\mathbb{R}, \mathbb{R}^2)$ satisfying $F(0) = 0$ and let $\omega_0 \in L^2$ and $\theta_0 \in L^2 \cap L^\infty$. Then we have :

$$\|v(t)\|_{LL} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2 \cap L^\infty} + \|\omega(t)\|_{LBMO}.$$

Proof. We use the following estimation for $\|v(t)\|_{LL}$ which proved in [2],

$$\|v(t)\|_{LL} \leq \|\omega(t)\|_{L^2} + \|\omega(t)\|_{B^0_{\infty,\infty}}$$
$$\leq \|\omega(t)\|_{L^2} + \|\omega(t)\|_{BMO}, \qquad (4.1)$$

where we have used in the last line the embedding $BMO \hookrightarrow B^0_{\infty,\infty}$. It remains then to estimate $\|\omega(t)\|_{L^2}$. For this, we use the first the equation of ω :

$$\partial_t\omega + v.\nabla\omega = \partial_1(F_2(\theta)) - \partial_2(F_1(\theta)), \qquad \omega(t=0) = \omega_0$$
$$= \grave{F_2}(\theta)\partial_1\theta - \grave{F_1}(\theta)\partial_2\theta$$

Taking the L^2 norm and applying Holder inequality, we get :

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \sum_{i=1}^{2}\int_{\mathbb{R}^2}\left\|\grave{F_i}\circ\theta(\tau)\right\|_{L^\infty}\|\nabla\theta(\tau)\|_{L^2}d\tau \qquad (4.2)$$

Now, since :

$$\left\|\grave{F_i}\circ\theta(t)\right\|_{L^\infty} = sup_{x,t}\left|\grave{F_i}(\theta(x,t))\right|,$$

then using the embedding $B^0_{\infty,\infty} \hookrightarrow \dot{B}^0_{\infty,\infty} \hookrightarrow L^\infty$ and Theorem 1.2 , we get :

$$\left\|\grave{F_i}\circ\theta(t)\right\|_{L^\infty} \leq sup_{|x|\leq C\|\theta\|_{L^\infty}}\left|\grave{F_i}(x)\right|$$
$$\leq sup_{|x|\leq\|\theta_0\|_{L^\infty}}\left|\grave{F_i}(x)\right|.$$

As $F \in C^1(\mathbb{R},\mathbb{R}^2)$, $\theta_0 \in L^2 \cap L^\infty$ and applying Theorem 1.2, yields:

$$\left\|\grave{F_i}\circ\theta(t)\right\|_{L^\infty} \leq sup_{|x|\leq\|\theta_0\|_{L^\infty}}\|\nabla F_i\|_{L^\infty} \leq C.$$

Then gives in (4.2), that :

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\nabla\theta\|_{L^1_tL^2} \qquad (4.3)$$

It remains then to estimate $\|\nabla\theta\|_{L^1_tL^2}$. For this purpose, we take the scalar product of the second equation of (1.1) with θ in L^2space. Then the incompressibility condition $div\, v = 0$ leads to the following energy estimate :

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2_{L^2} + 2\|\nabla\theta\|^2_{L^2} = 0.$$

Then :

$$\frac{d}{dt}\|\theta(t)\|_{L^2}^2 + 4\|\nabla\theta(t)\|_{L^2}^2 = 0.$$

Integrating in time, we get :

$$\|\theta(t)\|_{L^2}^2 + 4\|\nabla\theta\|_{L_t^1L^2}^2 = \|\theta_0\|_{L^2}^2.$$

Therefore,

$$\|\nabla\theta\|_{L_t^1L^2}^2 \leq \|\theta_0\|_{L^2}^2.$$

This implies that :

$$\|\nabla\theta\|_{L_t^1L^2} \leq \|\theta_0\|_{L^2}$$

This gives in (4.3) that :

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2}.$$

Plugging in (4.1), we obtain :

$$\|v(t)\|_{LL} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2} + \|\omega(t)\|_{BMO}.$$

Using the first property of Proposition 3.1, yields :

$$\|v(t)\|_{LL} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2\cap L^\infty} + \|\omega(t)\|_{LBMO}.$$

Proposition 4.2

Under the same hypothesis of Proposition 4.1, if in addition $\omega_0 \in LBMO \cap L^2$, then we have :

$$\|\omega(t)\|_{LBMO\cap L^2} \leq C_0 e^{C_0 t}.$$

Proof. Assume that ψ_t is the flow associated to the velocity v. Then we can write ω as :

$$\omega(t,x) = (\omega_0 \circ \psi_t^{-1})(x) = \omega_0\big(\psi_t^{-1}(x)\big) \qquad (4.4)$$

Since v is smooth then $\psi_t^{\pm1}$ is Lipschitzian for every $t \geq 0$. Then $\left\|\psi_t^{\pm1}\right\|_*$ is finite for every $t \geq 0$. Applying Theorem 3.2 to the equation (4.4), we get :

$$\|\omega(t)\|_{LBMO\cap L^2} \leq C\|\omega_0\|_{LBMO\cap L^2}\ln(1 + \|\psi_t^{-1}\|_*).$$

Using Proposition 2.1 to get :

$$\|\omega(t)\|_{LBMO\cap L^2} \lesssim \|\omega_0\|_{LBMO\cap L^2} \ln(1+\exp(\int_0^t \|v(\tau)\|_{LL} d\tau))$$

$$\leq C_0\left(1+\int_0^t \|v(\tau)\|_{LL} d\tau\right) \qquad (4.5)$$

Using Proposition 4.1, yields :

$$\|v(t)\|_{LL} \leq C_0\left(1+\int_0^t \|v(\tau)\|_{LL}\right) d\tau.$$

By Lemma 2.1 of Gronwall, we get :

$$\|v(t)\|_{LL} \leq C_0 e^{C_0 t}.$$

Plugging in (4.5), we get :

$$\|\omega(t)\|_{LBMO\cap L^2} \leq C_0 e^{C_0 t},$$

which is the desired result.

5. Conclusion :

We present the global weak solution for the Euler stratified system in two dimensional space with a function $F \in C^1(\mathbb{R},\mathbb{R}^2)$, such that $F(0)=0$ in the $LBMO$ space. We used the concept of the flow associated to the velocity and the L^2 estimate of the vorticity. Also, we applied some harmonic analysis for the function F.

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