On the Bernstein inequality

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Abstract

Chemin [1], *proved the inequality of Bernstein for any tempered distribution* u. *In this paper, we will extend its proof for a bloc dyadic* $\dot{\Delta}_q u$ *and* $S_q u$. *We will use the Fourier transform and apply the Yong inequality for convolution. In addition, we will use the techniques of analysis in frequency space.*

Keywords*: Dyadic decomposition, Littlewood-Paley operators, radial functions, space of Schwartz, Bernstein inequality.*

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1. Introduction

In this section, we recall the Young inequality, we define the dyadic decomposition of the full space $\mathbb{R}^d$ and recall the Littlewood-Paley operators.

The following inequality is well-known, can be found for example in [3].

Lemma 1.1: (Young convolution inequality)

For any two functions f and g, such that $f \in L^c$ and $g \in L^a$ and any constants $(a, b, c) \in [1, \infty]^3$, such that

$$1 + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}.$$

Then we have $f * g \in L^b$ and

$$\|f * g\|_{L^b} \leq C\|f\|_{L^c}\|g\|_{L^a}, \quad C \text{ is a constant}.$$

We can conclude immediate the following result, we refer to [1], [2].

Lemma 1.2 :

For every function $f \in \mathcal{S}$, where $\mathcal{S}$ is the space of Schwartz such that $f \in L^1 \cap L^\infty$ and for every $1 < c < \infty$, then we have $f \in L^c$ and $(1 + |\cdot|^2)^d \partial^\alpha f$ is bounded.

We recall the Littlewood-Paley operators see [1], [4] and [6] for more details.

Definition 1.3:

There exist two non-negative radial functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d/\{0\})$ such that

$$\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d,$$

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^d/\{0\},$$

$$|p-q| \geq 2 \Rightarrow supp\, \varphi(2^{-p}.) \cap supp\, \varphi(2^{-q}.) = \phi,$$

$$q \geq 1 \Rightarrow supp\, \chi \cap upp\, \varphi(2^{-q}.) = \phi.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\bar{h} = \mathcal{F}^{-1}\chi$, the frequency localization inhomogeneous operators Δ_q and S_q are defined by

$$\Delta_q f = \varphi(2^{-q}D)f, \qquad S_q f = \chi(2^{-q}D)f$$

$$\Delta_{-1} f = S_0 f, \qquad \Delta_q f = 0 \quad for\ q \leq -2.$$

And the frequency localization homogeneous operators $\dot{\Delta}_q$ and $\dot{S}_q$ are defined by

$$\dot{\Delta}_q f = \varphi(2^{-q}D)f, \qquad \dot{S}_q f = \chi(2^{-q}D)f$$

We notice that $\Delta_q = \dot{\Delta}_q, \forall q \in \mathbb{N}$ and S_q coincides with $\dot{S}_q$ on tempered distributions modulo polynomials.

From the definition of the operator Δ_q , we can write ([3], [5]),

$$u = \sum_q \Delta_q u$$

2. Bernstein inequality

In this section, we will prove a Bernstein inequality for a tempered distribution u with a bloc dyadic $\dot{\Delta}_q$ and S_q which is the main result of this paper.

Lemma 2.1: (Bernstein Lemma) There exists a constant $C > 0$ such that for every $q \in \mathbb{Z}, k \in \mathbb{N}$ and for every tempered distribution u we have

$$sup_{|\alpha|=k} \left\| \partial^\alpha S_q u \right\|_{L^b} \leq C^k 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \left\| S_q u \right\|_{L^a} \quad , b \geq a \geq 1 \ldots\ldots (1)$$

$$C^{-k}2^{qk}\|\dot{\Delta}_q u\|_{L^a} \le sup_{|\alpha|=k}\|\partial^\alpha \dot{\Delta}_q u\|_{L^a} \le C^k 2^{qk}\|\dot{\Delta}_q u\|_{L^a} \dots\dots (2)$$

Proof:

(1) Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ Such that $\varphi = 1$ in the neighborhood of the ball of center 0 and radius r_1. Let also $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^d)$ Such that $\tilde{\varphi} \equiv 1$ in the neighborhood of φ, then we have

$$S_q u = \tilde{\varphi}(2^{-q}D)S_q u.$$

It is clear that

$$S_q u = \mathcal{F}^{-1}\left(\tilde{\varphi}(2^{-q}D)\mathcal{F}(S_q u)\right) = \mathcal{F}^{-1}(\tilde{\varphi}(2^{-q}D)) * S_q u.$$

Using the Fourier transform with a simple calculation, we get

$$\mathcal{F}^{-1}(\tilde{\varphi}(2^{-q}D)) = \int \tilde{\varphi}(2^{-q}\xi)e^{ix\xi}d\xi = 2^{qd}\int \tilde{\varphi}(\xi)e^{ix2^q\xi}d\xi$$
$$= 2^{qd}\mathcal{F}^{-1}(\tilde{\varphi}(\xi)) \coloneqq 2^{qd}h(2^q x).$$

This gives that

$$S_q u = 2^{qd}h(2^q\,\cdot) * S_q u.$$

Therefore

$$\partial^\alpha S_q u = 2^{q(d+|\alpha|)}\partial^\alpha h(2^q\,\cdot) * S_q u \dots\dots (3)$$

Taking the L^b norm of (3) and applying Young inequality for convolution (Lemma 1.1), we find with $\left(\frac{1}{b} + 1 = \frac{1}{c} + \frac{1}{a}\right)$, that

$$\|\partial^\alpha S_q u\|_{L^b} \le 2^{q(d+|\alpha|)}\|\partial^\alpha h(2^q\,\cdot)\|_{L^c}\|S_q u\|_{L^a},$$
$$\le 2^{q(d+|\alpha|)}2^{-q\frac{d}{c}}\|\partial^\alpha h\|_{L^c}\|S_q u\|_{L^a}$$
$$\le 2^{q\left(|\alpha|+d\left(1-\frac{1}{c}\right)\right)}\|\partial^\alpha h\|_{L^c}\|S_q u\|_{L^a}$$
$$\le 2^{q\left(|\alpha|+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\|\partial^\alpha h\|_{L^c}\|S_q u\|_{L^a}.$$

Therefore

$$sup_{|\alpha|=k}\left\|\partial^{\alpha}S_q u\right\|_{L^b} \leq 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\|\partial^{\alpha}h\|_{L^c}\left\|S_q u\right\|_{L^a} \dots\dots (4)$$

It is enough to prove that $\|\partial^{\alpha}h\|_{L^c} \leq C^k$. For this purpose, we use Lemma 1.2, then we have

$$\|\partial^{\alpha}h\|_{L^c} \leq \|\partial^{\alpha}h\|_{L^1} + \|\partial^{\alpha}h\|_{L^\infty} \dots\dots (5)$$

Now since $h = \mathcal{F}^{-1}\varphi$ and $\varphi \in \mathcal{C}_0^{\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{S}$, where $\mathcal{S}$ is the space of Schwartz, then we have $h \in \mathcal{S}$, this gives by using Lemma 1.2, that h is bounded and $(1+|\cdot|^2)^d\partial^{\alpha}h$ is also bounded.

$$\|\partial^{\alpha}h\|_{L^1} = \int |\partial^{\alpha}h| \leq \int (1+|\cdot|^2)^{-d}(1+|\cdot|^2)^{d}|\partial^{\alpha}h|$$

$$\leq \left\|(1+|\cdot|^2)^{-d}\right\|_{L^1}\left\|(1+|\cdot|^2)^{d}\partial^{\alpha}h\right\|_{L^\infty}$$

$$\leq C\left\|(1+|\cdot|^2)^{d}\partial^{\alpha}h\right\|_{L^\infty} \dots\dots (6)$$

Also

$$\|\partial^{\alpha}h\|_{L^\infty} = sup_x|\partial^{\alpha}h(x)| \leq sup_x(1+|\cdot|^2)^{d}|\partial^{\alpha}h|$$

$$\leq C\left\|(1+|\cdot|^2)^{d}\partial^{\alpha}h\right\|_{L^\infty} \dots\dots (7)$$

Putting together (6) and (7) in (5), we get

$$\|\partial^{\alpha}h\|_{L^c} \leq C^2\left\|(1+|\cdot|^2)^{d}\partial^{\alpha}h\right\|_{L^\infty} \leq C^k, \qquad k \in \mathbb{N}.$$

This gives in (4), that

$$sup_{|\alpha|=k}\left\|\partial^{\alpha}S_q u\right\|_{L^b} \leq C^k\, 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)}\left\|S_q u\right\|_{L^a}.$$

This proves (1) of Lemma 2.1.

Proof of (2) of Lemma 2.1:

Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^d)$ Such that $\tilde{\varphi} = 1$ in the neighborhood of φ. Then we have

$$\dot{\Delta}_q u = \tilde{\varphi}(2^{-q}D)\dot{\Delta}_q u.$$

It is clear that

$$\mathcal{F}\left(\dot{\Delta}_q u\right) = \tilde{\varphi}(2^{-q}\xi)\mathcal{F}\left(\dot{\Delta}_q u\right),$$

and thus

$$\dot{\Delta}_q u(x) = \mathcal{F}^{-1}\left(\tilde{\varphi}(2^{-q}\xi)\mathcal{F}\left(\dot{\Delta}_q u\right)(\xi)\right) \dots\dots (8)$$

Where,

$$\tilde{\varphi}(2^{-q}\xi) = \sum_{|\alpha|=k} (i\xi)^\alpha |\xi|^{-2k} (-i\xi)^\alpha \tilde{\varphi}(2^{-q}\xi)$$

Putting this last inequality in (8), we get

$$\dot{\Delta}_q u(x) = \sum_{|\alpha|=k} \mathcal{F}^{-1}\left((i\xi)^\alpha |\xi|^{-2k} (-i\xi)^\alpha \tilde{\varphi}(2^{-q}\xi)\mathcal{F}\left(\dot{\Delta}_q u\right)(\xi)\right)$$

$$= \sum\nolimits_{|\alpha|=k} \mathcal{F}^{-1}\left((i\xi)^\alpha |\xi|^{-2k} \tilde{\varphi}(2^{-q}\xi)\mathcal{F}\left(\partial^\alpha \dot{\Delta}_q u\right)(\xi)\right)$$

$$= \sum_{|\alpha|=k} \mathcal{F}^{-1}\left((i\xi)^\alpha |\xi|^{-2k} \tilde{\varphi}(2^{-q}\xi)\right) * \partial^\alpha \dot{\Delta}_q u(x),$$

where,

$$\mathcal{F}^{-1}\left((i\xi)^\alpha |\xi|^{-2k} \tilde{\varphi}(2^{-q}\xi)\right) = \int \frac{(i\xi)^\alpha}{|\xi|^{2k}} \tilde{\varphi}(2^{-q}\xi) e^{ix\xi} d\xi$$

$$= \int \frac{(i2^q\xi)^\alpha}{|2^q\xi|^{2k}} \tilde{\varphi}(\xi) e^{ix2^{qd}\xi} d\xi$$

$$= 2^{q(d+|\alpha|-2k)} \int \frac{(i\xi)^\alpha}{|\xi|^{2k}} \tilde{\varphi}(\xi) e^{ix2^{qd}\xi} d\xi = 2^{q(d+|\alpha|-2k)} h_k(2^q x),$$

where

$$h_k(2^q x) = \int \frac{(i\xi)^\alpha}{|\xi|^{2k}} \tilde{\varphi}(\xi) e^{ix2^{qd}\xi} d\xi.$$

Then

$$\dot{\Delta}_q u(x) = 2^{q(d+|\alpha|-2k)} h_k(2^q \cdot) * \partial^\alpha \dot{\Delta}_q u.$$

This gives by Young inequality for convolution, that

$$\left\|\dot{\Delta}_q u\right\|_{L^a} \leq 2^{q(d+|\alpha|-2k)} \|h_k(2^q \cdot)\|_{L^1} \left\|\partial^\alpha \dot{\Delta}_q u\right\|_{L^a}.$$

We have

$$\|h_k(2^q \cdot)\|_{L^1} = \int |h_k(2^q x)| dx = \int |h_k(y)| 2^{-qd} dy = 2^{-qd} \|h_k\|_{L^1}.$$

We recall that, $h = \mathcal{F}^{-1}\varphi$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}$, where $\mathcal{S}$ is the space of Schwartz, then we have $h \in \mathcal{S}$, this gives by using Lemma 1.2, that h is bounded and $(1 + |\cdot|^2)^d \partial^\alpha h$ is also bounded. Therefore

$$\|h\|_{L^1} \leq C^k.$$

Thus

$$\left\|\dot{\Delta}_q u\right\|_{L^a} \leq 2^{q(|\alpha|-2k)} C^k \left\|\partial^\alpha \dot{\Delta}_q u\right\|_{L^a}.$$

Thus

$$sup_{|\alpha|=k} \left\|\partial^{\dot{\alpha}} \Delta_q u\right\|_{L^a} \geq C^{-k} 2^{qk} \left\|\dot{\Delta}_q u\right\|_{L^a}.$$

This is the desired result.

3. Conclusion

Bernstein inequality has been proved by J-Y-Chemin for any tempered distribution u. In this paper, we proved the inequality for a bloc dyadic $\dot{\Delta}_q u$ and $S_q u$. Our results show a strong support for the effect of mathematics and physical applications, for example, in the non-linear Navier-Stokes and Euler Boussinesq equations and the quasi-geostrophic equation.

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