On the Bernstein inequality

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Abstract
Chemin [1], proved the inequality of Bernstein for any tempered distribution $u$. In this paper, we will extend its proof for a bloc dyadic $\hat{\Delta}_q u$ and $S_q u$. We will use the Fourier transform and apply the Yong inequality for convolution. In addition, we will use the techniques of analysis in frequency space.

Keywords: Dyadic decomposition, Littlewood-Paley operators, radial functions, space of Schwartz, Bernstein inequality.

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1. Introduction

In this section, we recall the Young inequality, we define the dyadic decomposition of the full space $\mathbb{R}^d$ and recall the Littlewood-Paley operators.

The following inequality is well-known, can be found for example in [3].

**Lemma 1.1:** (Young convolution inequality)

For any two functions $f$ and $g$, such that $f \in L^c$ and $g \in L^a$ and any constants $(a, b, c) \in [1, \infty]^3$, such that

$$1 + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

Then we have $f \ast g \in L^b$ and

$$\|f \ast g\|_{L^b} \leq C\|f\|_{L^c}\|g\|_{L^a}, \quad C \text{ is a constant.}$$

We can conclude immediate the following result, we refer to [1], [2].

**Lemma 1.2:**

For every function $f \in S$, where $S$ is the space of Schwartz such that $f \in L^1 \cap L^\infty$ and for every $1 < c < \infty$, then we have $f \in L^c$ and $(1 + |\cdot|^2)^d \partial^a f$ is bounded.

We recall the Littlewood-Paley operators see [1], [4] and [6] for more details.

**Definition 1.3:**

There exist two non-negative radial functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d/\{0\})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^d.$$
\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^d / \{0\},
\]

\[|p - q| \geq 2 \Rightarrow \text{supp } \varphi(2^{-p}) \cap \text{supp } \varphi(2^{-q}) = \emptyset, \quad q \geq 1 \Rightarrow \text{supp } \chi \cap \text{upp } \varphi(2^{-q}) = \emptyset.\]

Let \( h = \mathcal{F}^{-1} \varphi \) and \( \bar{h} = \mathcal{F}^{-1} \chi \), the frequency localization inhomogeneous operators \( \Delta_q \) and \( S_q \) are defined by
\[
\Delta_q f = \varphi(2^{-q} D) f, \quad S_q f = \chi(2^{-q} D) f
\]
\[
\Delta_{-1} f = S_0 f, \quad \Delta_q f = 0 \quad \text{for } q \leq -2.
\]

And the frequency localization homogeneous operators \( \hat{\Delta}_q \) and \( \hat{S}_q \) are defined by
\[
\hat{\Delta}_q f = \varphi(2^{-q} D) f, \quad \hat{S}_q f = \chi(2^{-q} D) f
\]

We notice that \( \Delta_q = \hat{\Delta}_q, \forall q \in \mathbb{N} \) and \( S_q \) coincides with \( \hat{S}_q \) on tempered distributions modulo polynomials.

From the definition of the operator \( \Delta_q \), we can write ([3], [5]),
\[
u = \sum_q \Delta_q u
\]

2. Bernstein inequality

In this section, we will prove a Bernstein inequality for a tempered distribution \( u \) with a bloc dyadic \( \hat{\Delta}_q \) and \( S_q \) which is the main result of this paper.

Lemma 2.1: (Bernstein Lemma) There exists a constant \( C > 0 \) such that for every \( q \in \mathbb{Z}, k \in \mathbb{N} \) and for every tempered distribution \( u \) we have
\[
\sup_{|\alpha| = k} \| \partial^\alpha S_q u \|_{L^b} \leq C^k 2^{q \left( k + d \left( \frac{1}{a} \frac{1}{b} \right) \right)} \| S_q u \|_{L^a}, \quad b \geq a \geq 1 \ldots (1)
\]
\[ C^{-k}2^{qk}\|\hat{\Delta}q u\|_L^a \leq \sup_{|\alpha|=k} \|\partial^\alpha \hat{\Delta}q u\|_L^a \leq C^{-k}2^{qk}\|\hat{\Delta}q u\|_L^a \quad \cdots (2) \]

**Proof:**

(1) Let \( \varphi \in C_0^\infty(\mathbb{R}^d) \) such that \( \varphi = 1 \) in the neighborhood of the ball of center 0 and radius \( r_1 \). Let also \( \tilde{\varphi} \in C_0^\infty(\mathbb{R}^d) \) such that \( \tilde{\varphi} \equiv 1 \) in the neighborhood of \( \varphi \), then we have

\[ S_q u = \tilde{\varphi}(2^{-q}D)S_q u. \]

It is clear that

\[ S_q u = \mathcal{F}^{-1}(\tilde{\varphi}(2^{-q}D)\mathcal{F}(S_q u)) = \mathcal{F}^{-1}(\tilde{\varphi}(2^{-q}D)) \ast S_q u. \]

Using the Fourier transform with a simple calculation, we get

\[
\mathcal{F}^{-1}(\tilde{\varphi}(2^{-q}D)) = \int \tilde{\varphi}(2^{-q}x)e^{ix\xi}d\xi = 2^{qd} \int \tilde{\varphi}(\xi)e^{ix2^q\xi}d\xi = 2^{qd}\mathcal{F}^{-1}(\tilde{\varphi}(\xi)) := 2^{qd}h(2^qx).
\]

This gives that

\[ S_q u = 2^{qd} h(2^q \cdot) \ast S_q u. \]

Therefore

\[ \partial^\alpha S_q u = 2^{q(d+|\alpha|)} \partial^\alpha h(2^q \cdot) \ast S_q u \quad \cdots (3) \]

Taking the \( L^b \) norm of (3) and applying Young inequality for convolution (Lemma 1.1), we find with \( \left( \frac{1}{b} + 1 = \frac{1}{c} + \frac{1}{a} \right) \), that

\[
\|\partial^\alpha S_q u\|_{L^b} \leq 2^{q(d+|\alpha|)} \|\partial^\alpha h(2^q \cdot)\|_{L^c} \|S_q u\|_{L^a},
\]

\[
\leq 2^{q(d+|\alpha|)} 2^{-q\frac{d}{c}} \|\partial^\alpha h\|_{L^c} \|S_q u\|_{L^a}
\]

\[
\leq 2^{q\left( |\alpha|+d\left(1-\frac{1}{c}\right)\right)} \|\partial^\alpha h\|_{L^c} \|S_q u\|_{L^a}
\]

\[
\leq 2^{q\left( |\alpha|+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|\partial^\alpha h\|_{L^c} \|S_q u\|_{L^a}.
\]
Therefore
\[
\sup_{|\alpha|=k} \|\partial^\alpha S_qu\|_{L^b} \leq 2^q \left( k + d \left( \frac{1}{\alpha} \right) \right) \||\partial^\alpha h\|_{L^c} \||S_qu\|_{L^a} \ldots \ldots (4)
\]

It is enough to prove that \(\|\partial^\alpha h\|_{L^c} \leq C^k\). For this purpose, we use Lemma 1.2, then we have
\[
\|\partial^\alpha h\|_{L^c} \leq \|\partial^\alpha h\|_{L^1} + \|\partial^\alpha h\|_{L^\infty} \ldots \ldots (5)
\]

Now since \(h = F^{-1} \varphi\) and \(\varphi \in C_0^\infty(\mathbb{R}^d) \hookrightarrow S\), where \(S\) is the space of Schwartz, then we have \(h \in S\), this gives by using Lemma 1.2, that \(h\) is bounded and \((1 + |\cdot|^2)^d \partial^\alpha h\) is also bounded.
\[
\|\partial^\alpha h\|_{L^1} = \int |\partial^\alpha h| \leq \int (1 + |\cdot|^2)^{-d} (1 + |\cdot|^2)^d |\partial^\alpha h|
\leq \|(1 + |\cdot|^2)^{-d}\|_{L^1} \|(1 + |\cdot|^2)^d\|_{L^\infty} \ldots \ldots (6)
\]

Also
\[
\|\partial^\alpha h\|_{L^\infty} = \sup_x |\partial^\alpha h(x)| \leq \sup_x (1 + |\cdot|^2)^d |\partial^\alpha h|
\leq C \|(1 + |\cdot|^2)^d \partial^\alpha h\|_{L^\infty} \ldots \ldots (7)
\]

Putting together (6) and (7) in (5), we get
\[
\|\partial^\alpha h\|_{L^c} \leq C^2 \|(1 + |\cdot|^2)^d \partial^\alpha h\|_{L^\infty} \leq C^k, \quad k \in \mathbb{N}.
\]

This gives in (4), that
\[
\sup_{|\alpha|=k} \|\partial^\alpha S_qu\|_{L^b} \leq C^k 2^q \left( k + d \left( \frac{1}{\alpha} \right) \right) \||S_qu\|_{L^a}.
\]

This proves (1) of Lemma 2.1.
Proof of (2) of Lemma 2.1:

Let \( \tilde{\phi} \in C_0^\infty(\mathbb{R}^d) \) such that \( \tilde{\phi} = 1 \) in the neighborhood of \( \phi \). Then we have

\[ \dot{\Delta}_q u = \tilde{\phi}(2^{-q}D)\dot{\Delta}_q u. \]

It is clear that

\[ F(\dot{\Delta}_q u) = \tilde{\phi}(2^{-q}\xi)F(\dot{\Delta}_q u), \]

and thus

\[ \dot{\Delta}_q u(x) = F^{-1} \left( \tilde{\phi}(2^{-q}\xi)F(\dot{\Delta}_q u)(\xi) \right) \ldots \quad (8) \]

Where,

\[ \tilde{\phi}(2^{-q}\xi) = \sum_{|\alpha|=k} (i\xi)\alpha|\xi|^{-2k}(-i\xi)\alpha \tilde{\phi}(2^{-q}\xi) \]

Putting this last inequality in (8), we get

\[ \dot{\Delta}_q u(x) = \sum_{|\alpha|=k} F^{-1} \left( (i\xi)\alpha|\xi|^{-2k}(-i\xi)\alpha \tilde{\phi}(2^{-q}\xi)F(\partial^\alpha \dot{\Delta}_q u)(\xi) \right) \]

\[ = \sum_{|\alpha|=k} F^{-1} \left( (i\xi)\alpha|\xi|^{-2k} \tilde{\phi}(2^{-q}\xi)F(\partial^\alpha \dot{\Delta}_q u)(\xi) \right) \]

\[ = \sum_{|\alpha|=k} F^{-1} \left( (i\xi)\alpha|\xi|^{-2k} \tilde{\phi}(2^{-q}\xi) \right) \ast \partial^\alpha \dot{\Delta}_q u(x), \]

where,

\[ F^{-1} \left( (i\xi)\alpha|\xi|^{-2k} \tilde{\phi}(2^{-q}\xi) \right) = \int \frac{(i\xi)^\alpha}{|\xi|^{2k}} \tilde{\phi}(2^{-q}\xi) e^{ix\xi} d\xi \]

\[ = \int \frac{(i2^q\xi)^\alpha}{|2^q\xi|^{2k}} \tilde{\phi}(\xi) e^{ix2^{qd}\xi} d\xi \]

\[ = 2^q(d+|\alpha|-2k) \int \frac{(i\xi)^\alpha}{|\xi|^{2k}} \tilde{\phi}(\xi) e^{ix2^qd\xi} d\xi = 2^q(d+|\alpha|-2k) h_k(2^q\xi), \]
where
\[ h_k(2^q x) = \int \frac{(i\xi)\alpha}{|\xi|^{2k}} \hat{\phi}(\xi) e^{ix2^q\xi} d\xi. \]

Then
\[ \hat{\Delta}_q u(x) = 2^{q(d+|\alpha|-2k)} h_k(2^q \cdot) * \partial^\alpha \hat{\Delta}_q u. \]

This gives by Young inequality for convolution, that
\[ \|\hat{\Delta}_q u\|_{L^a} \leq 2^{q(d+|\alpha|-2k)} \|h_k(2^q \cdot)\|_{L^1} \|\partial^\alpha \hat{\Delta}_q u\|_{L^a}. \]

We have
\[ \|h_k(2^q \cdot)\|_{L^1} = \int |h_k(2^q x)| dx = \int |h_k(y)| 2^{-qd} dy = 2^{-qd} \|h_k\|_{L^1}. \]

We recall that, \( h = F^{-1} \varphi \) and \( \varphi \in C_0^\infty(\mathbb{R}^d) \leftrightarrow S \), where \( S \) is the space of Schwartz, then we have \( h \in S \), this gives by using Lemma 1.2, that \( h \) is bounded and \((1 + |\cdot|^2)^d \partial^\alpha h\) is also bounded. Therefore
\[ \|h\|_{L^1} \leq C^k. \]

Thus
\[ \|\hat{\Delta}_q u\|_{L^a} \leq 2^{q(|\alpha|-2k)} C^k \|\partial^\alpha \hat{\Delta}_q u\|_{L^a}. \]

Thus
\[ \sup_{|\alpha|=k} \|\partial^\alpha \hat{\Delta}_q u\|_{L^a} \geq C^{-k} 2^{qk} \|\hat{\Delta}_q u\|_{L^a}. \]

This is the desired result.

3. Conclusion

Bernstein inequality has been proved by J-Y-Chemin for any tempered distribution \( u \). In this paper, we proved the inequality for a block dyadic \( \hat{\Delta}_q u \) and \( S_q u \). Our results show a strong support for the effect of mathematics and physical applications, for example, in the non-linear Navier-Stokes and Euler Boussinesq equations and the quasi-geostrophic equation.
References: