linear Sequential Fractional Differential Equations

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Abstract

In This work presents the basic general theory for sequential linear fractional differential equations, involving the well-known Riemann-Liouville fractional operators.

We then introduce the Mittag-Leffler type function $e_{\alpha}^{\lambda x}$, which we will call α- exponential. This function is the product of a Mittag-Leffler function and a power function. This function allows us to directly obtain the general solution to homogeneous and non-homogeneous linear fractional differential equations with constant coefficients. This method is a variation of the usual one for the ordinary case.

المعادلات التفاضلية الكسرية الخطية المتسلسلة

الملخَّص:

في هذا العمل عرضنا التعاريف والنظريات الرئيسية الخاصة بالمعادلات التفاضلية الكسرية، والتي نحتاجها خلال هذا العمل، و قمنا أيضا بتطوير النظرية العامة الأساسية للمعادلات التفاضلية الكسرية الخطية المتسلسلة، والتي تتضمَّن معاملي ريمان – ليوفيل للكسور المعرفين، ثم قدمنا دالة Mittag–Leffler من نوع $e_{\alpha}^{\lambda(x-a)}$ والتي نسميها α–exponential وهذه الدالة هي نتاج دالة Mittag–Leffler ودالة القوى.

تسمح لنا دالة α–exponential بالحصول مباشرة على الحل العام للمعادلة التفاضلية الخطية المتجانسة، وغير المتجانسة ذات المعاملات الثابتة، وهذه الطريقة تختلف

عن الطريقة المعتادة للحالة العادية كما قمنا بوضع بعض الأمثلة التوضيحية خلال هذا العمل.

Keywords : Fractional Differential Equations, Caputo, Riemann-Liouville, Linear equation , a Mittag-Leffler function , .

1. Introduction

Fractional calculus was introduced on September 30, 1695. On that day, Leibniz wrote a letter to L'Hôpital , raising the possibility of generalizing the meaning of derivatives from integer order to noninteger order derivatives

L'Hôpital wanted to know the result for the derivative of order n = 1/2. Leibniz replied that "one day, useful consequences will be drawn" and, in fact, his vision became a reality. However, the study of noninteger order derivatives did not appear in the literature until 1819, when Lacroix presented a definition of fractional derivative based on the usual expression for the nth derivative of the power function (Lacroix 1819). Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., noninteger) differential operators were introduced: the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz (Hilfer 2000; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993) and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993) .

In 2010, an interesting perspective to the subject, unifying all mentioned notions of fractional derivatives and integrals, was introduced in Agrawal (2010) and later studied in Bourdin et al. (2014), Klimek and Lupa (2013), Odzijewicz et al. (2012a, b, 2013a, b, c). Precisely, authors considered general operators, which by choosing special kernels, reduce to the standard fractional operators.

It is known that the classical calculus provides a powerful tool for explaining and modelling many important dynamic processes in most applied sciences.

But experiments and reality teach us that there are many complex systems in nature and society with anomalous dynamics, such as charge transport in amorphous semiconductors, the spread of contaminants in underground water, relaxation in viscoelastic materials like polymers, the diffusion of pollution in the atmosphere, and many more.

In most of the above-mentioned cases, this kind of anomalous process has a complex macroscopic behavior, the dynamics of which cannot be characterised by classical derivative models. Nevertheless, a heuristic solution to the corresponding models of some of those processes can be frequently obtained using tools from statistical physics. For such an explanation, one must use some generalized concepts from classical physics such as fractional Brownian motion, the continuous time random walk (CTRW) method involving L´evy stable distributions (instead of Gaussian distributions), the generalized central limit theorem (instead of the classical central limit theorem), and non Markovian distributions which means non-local distributions (instead of the classical Markovian ones). From this approach it is also important to note that the anomalous behavior of many complex processes includes

multi-scaling in the time and space variables .

The above-mentioned tools have been used extensively during last 30 years. But the connection between these statistical models and certain fractional differential equations involving the fractional integral and derivative operators (Riemann-Liouville, Caputo, Liouville or Weyl, Riesz, etc.; see [11]) has only been formally established during the last 15 years; (see, for instance, [7], [6] [12], [10]).

We could ask to our self, what are the useful properties of these fractional calculus operators which help in the modelling of so many anomalous processes? From the point of view of the authors and from known experimental results, most of the processes associated with complex systems have non-local dynamics involving long-memory in time, and the fractional integral and fractional derivative operators do have some of those characteristics. Perhaps this is one of the reasons why these fractional calculus operators lose the above-mentioned useful properties of the ordinary derivative D.

This work is organized as follows. Sections 2 present some fractional operators and their main properties and introduce some types of Mittag-Leffler functions. In Section 3 we develop a general theory for sequential linear fractional differential equations, while in Section 4 we introduce upgraded direct method for solving the homogeneous and non-homogeneous case with constant coefficients, using the α-exponential function and certain fractional Green functions, including some illustrative examples.

2. Preliminaries

Definition 2.1. Let $R = (-\infty, \infty)$ and $R_+ = (0, \infty)$. We denote the space of function f by $C_r^n[0,T]$, where f satisfies $f: (0,T] \to R$ $(\forall\, T > O)$ and $t^r f^{(n)}(x) \in C[0,T]$ for $0 \le r < 1$. In particular, denote $C_r^0[0,T]$ by $C_r[0,T]$.

Definition 2.2 . [2] Let $\alpha \in R_+, f \in C_r[0,T], 0 \le r \le 1$. Then

$$(I_0^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \quad (x > a) \quad , \quad (1)$$

is called a fractional integral of order $\alpha\,(\alpha > 0)$ of the function f in the sense of Riemann– Liouville. In particular, we denote $I^0 f(x) = f(x)$.

Definition 2.3 . [2] Let $n-1 < \alpha \le n, n \in N, I^{n-\alpha} f \in C_\gamma^n[0,T]$ and $0 \le \gamma < 1$. Then

$$D_0^\alpha f(x) = D^n I^{n-\alpha} f(x), \qquad D^n = \frac{d^n}{dx^n} \quad (2)$$

is called the fractional derivative of order α of the function f in the sense of Riemann– Liouville.

Let us remember that, in general, when $\alpha, \beta \in R^+$, the operators $D_0^{\alpha+\beta}$ and $D_0^{\alpha} D_0^{\beta}$ are different. Also, as usual, we will use $AC\,([a,b])$ to refer to the set of absolutely continuous functions in $[a,b]$, and $AC^n([a,b])$ $(n \in N)$, for the set of functions f, such that there exist $(D^n)(f) = f^{(n}$ in $[a,b]$ and $f^{(n} \in AC$.

Definition 2.4 [2] Let $n-1 < \alpha < n, n \in N, I^{n-\alpha} \in C_r^n[0,T], 0 \le r < 1$. Then

$$ {}^{c}D_{0+}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k\right], $$

is called the Caputo fractional derivative of order α of the function f .

Remark When $\alpha = n$, we have ${}^{c}D_{0+}^{\alpha} f(t) = D_0^{\alpha} f(t) = D^n f(t)$.

Definition 2.5 . Let $\lambda, v \in \mathbb{C}$, $\alpha \in R^+$ and $a \in R$. We will call α-exponential function $e_{\alpha}^{\lambda(x-a)}$ the Mittag-Leffler type function

$$ e_{\alpha}^{\lambda(x-a)} = (x-a)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (x-a)^{k\alpha}}{\Gamma((k+1)\alpha)} \quad (x > a). \qquad (3) $$

Definition 2.6. Let $\alpha \in R^+$, $l \in N_0$, $a \in R$ and $\lambda = b + ic \in \mathbb{C}$. We will call $\varepsilon_{\alpha,l}^{\lambda x}$ the Mittag-Leffler type function

$$ e_{\alpha,l}^{\lambda(x-\alpha)} = (x-a)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(l+k)!}{\Gamma[(k+l+1)\alpha]} \frac{\left(\lambda(x-a)^{\alpha}\right)^k}{k!} \quad (x > a). \qquad (4) $$

Property 2.1. Let $n-1 \le \alpha < n$, $m-1 \le \beta < m$. If $f \in L_1(a,b)$ with $f_{m-\beta} \in AC^{m+1}([a,b])$ if $\alpha + \beta < n$ (or $f_{\{\alpha+\beta\}} \in AC^{\alpha+\beta}([a,b])$ if $\alpha + \beta > n$), where $f_{n-\alpha} = (I_{a+}^{n-\alpha} f)(x)$. Then we have the following rule

$$ \left(D_{a+}^{\alpha} D_{a+}^{\beta} f\right)(x) = \left(D_{a+}^{\alpha+\beta} f\right)(x) - \sum_{j=1}^{m} \left(D_{a+}^{\beta-j} f\right)(a+) \frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}, \ (5) $$

almost everywhere in $[a,b]$.

Property 2.2. Let $0 < \eta < 1$, $\left(D_{a+}^{\eta} K\right) \in L_1(a,b)$ with a suitable f (for example, $f \in C([a,b])$. Then we have

$D_a^{\eta} \int_a^x K(x-t)f(t)dt = \int_a^x [D_{a+}^{\eta} K(x-a)](t)f(x-t+a)dt + f(x)\lim_{a^+}[I_{a+}^{1-\eta}K(t-a)](x).$ (6)

As expected, a fractional differential equation of order αn is an equation such as

$$F\big(x, y(x), (D^{\alpha_1}y)(x), (D^{\alpha_2}y)(x), \dots, (D^{\alpha_n}y)(x)\big) = g(x), \qquad (7)$$

with $\alpha_1 < \alpha_2 < \cdots < \alpha_n, F(x, y_1, \dots, y_n)$ and $g(x)$ known real functions, $D^{\alpha k}$ $(k = 1,2, \dots, n)$ fractional differential operators and where $y(x)$ is the unknown function.

In 1993 Miller-Ross [8] introduced the so called sequential fractional derivative D^{α} in the following way

$$D^{\alpha} = D^{\alpha}, \quad (0 < \alpha \leq 1)$$

$$D^{k\alpha} = D^{\alpha}D^{(k-1)\alpha}, \quad (k = 2,3, \dots), \qquad (8)$$

where D^{α} is a fractional derivative.

A sequential fractional differential equation of order $n\alpha$ has the following relationship

$$F\big(x, y(x), (D^{\alpha}y)(x), (D^{2\alpha}y)(x), \dots, (D^{n\alpha}y)(x)\big) = g(x) \qquad (9)$$

Let $D^{\alpha} = D_{a+}^{\alpha}$ be the Riemann-Liouville fractional derivative. Then, taking into account Property 1, we can obtain the relation between $D_{a+}^{n\alpha}$ and $D_{a+}^{n\alpha}$.

When $n = 2$ such relation is given by

$$(D_{a+}^{\alpha}\, y)(x) = D_{a+}^{2\alpha}\left[y(x) - (I_{a+}^{1-\alpha}\, y)(a+)\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\right], \qquad (10)$$

On the other hand, if $\alpha = \frac{n}{p}$ $(n, p \in N)$ and $y(x)$ is a continuous real function defined in $[a, b]$, that is $y \in ([a, b])$, we can deduce from Property 1 the important property: $(D^n\, y)(t) = \big(D_{a+}^{p\alpha}\, y\big)(t).$ $(t > a)$ (11)

In this work we study the linear sequential fractional differential equations of order $n\alpha$ which can be written in the following normalized form

$L_{n\alpha}(y) = [D_{a+}^{n\alpha} + \sum_{k=0}^{n-1} a_k(x) D_{a+}^{k\alpha}](y) = y^{(n\alpha} + \sum_{k=0}^{n-1} a_k(x) y^{(k\alpha} = f(x),$ (12)

where $\{a_k(x)\}_{k=0}^{n-1}$ are continuous real functions defined in an interval $[a,b] \subset R$ and $f(x) \in C([a,b])$ or $f(x) \in C((a,b])$.

The existence and uniqueness of solutions to the Cauchy type problem for fractional differential equation (12) was established in [3], [4], and [10] for different kinds of functional spaces. We present below two of the theorems which will be used in this work.

Theorem 2.1. Let $x_0 \in (a,b) \subset R$ and $\{y_0^k\}_{k=0}^{n-1} \in R^n$. Let $f(x)$ and $\{a_k(x)\}_{k=0}^{n-1}$ be continuous real functions in $[a,b]$. Then there exists a unique continuous function $y(x)$ defined in $(a,b]$ which is a solution to the Cauchy type problem

$[L_{n\alpha}(y)](x) = f(x)$ (13)

$(D_{a+}^{k\alpha}\, y)(x_0) = y^{(k\alpha}(x_0) = y_0^k \quad (k = 0,1,\dots,n-1),$ (14)

Moreover, this solution $y(x)$ satisfies

$\lim_{x\to a+}(x-a)^{1-\alpha}\, y(x) < \infty$, (15)

and

$(I_{a+}^{1-\alpha} y)(x) < \infty.$ (16)

We denote with $C_\gamma([a,b])$ $(\gamma \in R)$ the Banach space

$C_\gamma([a,b]) = \{g(x) \in C([a,b]):\ \|g\|c_\gamma = \|(x-a)^\gamma\, g(x)\|c < \infty\},$ (17)

In particular $C_0([a,b]) = C([a,b])$.

Theorem 2.2. Let $\{a_k(x)\}_{k=0}^{n-1}$ be continuous functions in $[a,b], f \in C_{1-\alpha}([a,b])$ and $\{b_k\}_{k=0}^{n-1} \in R^n$. Then there exists a unique continuous function $y(x)$ defined in $(a,b]$ which is a solution to the linear sequential fractional differential equation of order $n\alpha$

$[L_{n\alpha}(y)](x) = f(x)$ (18)

And such that

$\lim_{x\to a+}(x-a)^{1-\alpha}\,(D_{a+}^{k\alpha}\, y)(x) = b_k$ (19)

or such that $(I_{a+}^{1-\alpha}\, D_{a+}^{k\alpha}\, y)(a+) = b_k.$ (20)

Corollary 2.1. Let $x_0 \in (a,b]$, $(or\ x_0 = a)$. Let $\{a_k(x)\}_{k=0}^{n-1}$ be continuous real functions defined in $(a,b]$ and such that $(x -$

$a)^{1-\alpha}\, a_k(x)|_{x=a} < \infty, \forall\, k = 1,2, \dots, n.$ The homogeneous linear sequential fractional differential equation

$$[L_{n\alpha}(y)](x) = 0 \qquad (21)$$

has $y(x) = 0$ as the unique solution in $(a, b]$, satisfying the initial conditions

$$y^{(j\alpha}(x_0) = 0 \quad \left(\left[(x-\alpha)^{1-\alpha} y^{(k\alpha}(x)\right]_{x=a+} = 0\right) \qquad (k = 0,1, \dots, n-1).$$

3-General theory for linear fractional differential equations

In this section we study the solutions to a homogeneous linear sequential fractional differential equation

$$L_{n\alpha}(y) = [D_{a+}^{n\alpha} + \textstyle\sum_{k=0}^{n-1} a_k(x)\, D_{a+}^{k\alpha}](y) = y^{(n\alpha} + \sum_{k=0}^{n-1} a_k(x)\; y^{(k\alpha} = 0, \qquad (22)$$

where $\{a_k(x)\}_{k=0}^{n-1}$ are continuous real functions in $[a, b]$ and $[D_{a+}^{n\alpha}](y) = y^{(n\alpha}$ is the sequential Riemann-Liouville fractional derivative.

Definition 3.1. As usual, a fundamental set of solutions to equation (22) in some interval $V \subset [a, b]$ is a set of n functions linearly independent in V,which are solution to (22).

Definition 3.2. The α-Wronskian of the n functions $\{u_k(x)\}_{k=1}^{n}$, which admit iterated fractional derivatives up to order $(n-1)\alpha$ in some interval $V \subset [a, b]$, refers to the following determinant

$$|W_\alpha(u_1, \dots, u_n)(x)| = \begin{vmatrix} u_1(x) & u_2(x) & \dots & \dots & u_n(x) \\ u_1^{(\alpha}(x) & u_2^{(\alpha}(x) & \dots & \dots & u_n^{(\alpha}(x) \\ u_1^{(2\alpha}(x) & u_2^{(2\alpha}(x) & \dots & \dots & u_n^{(2\alpha}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{((n-1)\alpha}(x) & u_2^{((n-1)\alpha}(x) & \dots & \dots & u_n^{((n-1)\alpha}(x) \end{vmatrix}. \qquad (23)$$

To simplify the notation, this will be represented by $|W_\alpha(x)| = |W_\alpha(u_1, \dots, u_n)(x)|$. We will use $W_\alpha(x)$ for the corresponding Wronskian matrix.

Theorem 3.1. Let $\{u_k(x)\}_{k=1}^{n}$ be a family of functions with sequential fractional derivatives up to order $(n-1)\alpha$ in $(a,b]$ and such that, if $j=1,2,\dots,n$ and $k=0,1,\dots,n-1$

$$\lim_{x\to a+}\left[(x-a)^{1-\alpha}\ u_j^{(k\alpha}(x)\right] < \infty\ . \qquad (24)$$

If the functions $\left\{(x-a)^{1-\alpha}\ u_j(x)\right\}_{j=1}^{n}$ are linearly dependent in $[a,b]$, it follows that for all $x\in[a,b]$

$$(x-a)^{n-n\alpha}\ |W_\alpha(x)| = 0. \qquad (25)$$

We can complete the above result, as in the ordinary case, with the following theorem .

Theorem 3.2. Let $\{u_k(x)\}_{k=1}^{n}$ be a solution family of functions to equation (22) in $(a,b]$ which satisfies

$$\lim_{x\to a+}\left[(x-a)^{1-\alpha}\ \ u_j(x)\right] < \infty \qquad (j=1,2,\dots,n).$$

Then the functions

$$\left\{(x-a)^{1-\alpha}\ u_j(x)\right\}_{j=1}^{n}$$

are linearly dependent in $[a,b]$ if, and only if, there exists an $x_0\in[a,b]$ such that

$$[(x-a)^{n-n\alpha}\ |W_\alpha(x)|]_{x=x_0} = 0 \qquad (26)$$

From the above theorem we can always find, in a way similar to the ordinary case , a fundamental set of solutions for equation (22) in some interval $V\subset[a,b]$.

Usually, the general solution to a non-homogeneous linear sequential fractional differential equation

$$L_{n\alpha}(y) = f(x)\ . \qquad (27)$$

will be given as in the following proposition:

Proposition 3.1 . If $y_p(x)$ is a particular solution to (27) and $y_h(x)$ is a general solution to the corresponding homogeneous equation

$$L_{n\alpha}(y) = 0. \qquad (28)$$

That is,

$$y_h(x) = \sum_{k=1}^{n} c_k u_k(x), \qquad (29)$$

with $\{c_k\}_{k=1}^{n}$ arbitrary real constants and $\{u_k(x)\}_{k=1}^{n}$ a fundamental set of (28), then a general solution to the non-homogeneous equation (27) is

$$y_g(x) = y_h(x) + y_p(x), \qquad (30)$$

A general theory , similar to the above, can be established for the Caputo fractional derivative $D^a \equiv {}^{C}D_{a+}^{\alpha}$, which was introduced by Caputo in 1969, see, for instance, [1].

$$\left({}^{C}D_{a+}^{\alpha}f\right)(x) = (I_{a+}^{n-\alpha}\, D^n\, f)(x) \quad (x > a \quad and \quad n = -[-\alpha]). \qquad (31)$$

Also it is usual to consider the following, more general, definition for the Caputo fractional derivative

$$\left({}^{C}D_{a+}^{\alpha}f\right)(x) = D_{a+}^{\alpha}\left[f(x) - \sum_{j=0}^{n-1} f^{(j)}(a+)\frac{(x-a)^j}{j!}\right], \qquad (32)$$

which shows the close connection between the Caputo and the RiemannLiouville derivatives.

4-Linear sequential fractional differential equations with constant coefficients

In this section we present a direct method for obtaining the explicit general solution to a linear sequential fractional differential equation with constant coefficients, such as

$$L_{n\alpha}(y) = [D_{a+}^{n\alpha} + \sum_{k=0}^{n-1} a_k\, D_{a+}^{k\alpha}](y) = f(x), \qquad (33)$$

where a and $\{a_k(x)\}_{k=0}^{n-1}$ are real constants and $D_{a+}^{k\alpha}$ is the Riemann-Liouville sequential fractional derivative.

Several approaches have been developed for obtaining explicit solutions to some of these types of equations. The Laplace method was discussed by some authors, see, for instance, [8], [1], and [10] , but this approach is applicable only if $a = 0$. With the restriction 0, it is not possible to consider Cauchy type problems for equation (33) with conditions at $x = 0$. On the other hand, the direct method is very convenient for studying and solving boundary value problems associated with equation (33) which cannot be solved by the Laplace method. At the end, we will introduce a fractional Green function to obtain an explicit particular solution to the non-homogeneous equation (33).

Let us consider now the corresponding homogeneous equation to (33)

$$L_{n\alpha}(y) = [D_{a+}^{n\alpha} + \sum_{k=0}^{n-1} a_k \, D_{a+}^{k\alpha}](y) = 0. \quad (34)$$

As in the ordinary case, if we try to find solutions to (34) of the type $y(x) = e_\alpha^{\lambda(x-a)}$, it follows that

$$L_{n\alpha}\left(e_\alpha^{\lambda(x-a)}\right) = P_n(\lambda)e_\alpha^{\lambda(x-a)} \quad (35)$$

where

$$P_n(\lambda) = \lambda^n + \sum_{k=1}^{n-1} a_k\lambda^k , \quad (36)$$

is referred to as the characteristic polynomial associated with equation (34).

In the following it will be assumed that $\lambda \in \mathbb{C}$.

By the use of the properties of the **α-exponential** function, we obtain the following result .

Lemma 4.1. If λ is a root of characteristic polynomial (36), then

$$\frac{\partial}{\partial\lambda} L_{n\alpha}\left(e_\alpha^{\lambda(x-a)}\right) = L_{n\alpha} \frac{\partial}{\partial\lambda}\left(e_\alpha^{\lambda(x-a)}\right) \quad (37)$$

And

$$\frac{\partial^l}{\partial\lambda^l} e_\alpha^{\lambda(x-a)} = (x-a)^{l\alpha} \varepsilon_{\alpha,l}^{\lambda(x-a)} . \quad (38)$$

So we can connect the solution of the characteristic polynomial (36) with solutions of (34) as in the usual case

Theorem 4.1. Let $\{\lambda_j\}_{j=1}^k$ be all different real roots of the characteristic polynomial (36) , whose orders of multiplicity are $\{\mu_j\}_{j=1}^k$, respectively . Let $\{r_j, \bar{r}_j\}_{j=1}^p$ $(r_j = b_j + ic_j)$ be all distinct pairs of complex conjugate solutions of multiplicity $\{\sigma_j\}_{j=1}^p$, respectively, of (36). Then the union set of the sets

$$\cup_{m=1}^k\left\{(x-a)^{l\alpha} \varepsilon_{\alpha,l}^{\lambda(x-a)}\right\}_{l=1}^{\mu_m-1}, \quad (39)$$

$$\bigcup_{m=1}^{p}\left\{\sum_{j=0}^{\infty}(-1)^j\ \frac{c_m^{2j}}{(2j)!}\ (x-a)^{(2j+1)\alpha}\ \varepsilon_{\alpha,l+2j}^{b_m(x-a)}\right\}_{l=1}^{\sigma_m-1} \qquad (40)$$

and

$$\bigcup_{m=1}^{p}\left\{\sum_{j=0}^{\infty}(-1)^j\ \frac{c_m^{2j+1}}{(2j+1)!}\ (x-a)^{(2j+l+1)\alpha}\ \varepsilon_{\alpha,l+2j+1}^{b_m(x-a)}\right\}_{l=1}^{\sigma_m-1}, \qquad (41)$$

determines a fundamental system of solutions to fractional differential equation (34).

Note that only for the case where $a=0$ can operational methods such as the Laplace transform be applied to solve the problem of constant coefficients.

Example 4.1. Let us consider the equation

$$D_{a+}^{2\alpha}\ y+\lambda^2 y=0. \qquad (42)$$

Its characteristic equation is $P_2(x)=x^2+\lambda^2=(x-\lambda i)(x+\lambda i)$ and so the fundamental set of solutions to (42) is

$$\{\cos_\alpha[\lambda(x-a)],\ \sin_\alpha[\lambda(x-a)]\},$$

Where

$$\cos_\alpha[\lambda(x-a)]=\sum_{j=0}^{\infty}(-1)^j\ \lambda^{(2j+1)}\frac{(x-a)^{(j+1)2\alpha-1}}{\Gamma[(j+1)\alpha]} \qquad (43)$$

and

$$\sin_\alpha[\lambda(x-a)]=\sum_{j=0}^{\infty}(-1)^j\ \lambda^{2j}\frac{(x-a)^{(2j+1)\alpha-1}}{\Gamma[(2j+1)\alpha]}. \qquad (44)$$

These new functions $\sin_\alpha(x)$ and $\cos_\alpha(x)$ are a generalization of the usual $\cos(x)$ and $\sin(x)$.

Since now we know how to obtain the general solution to homogeneous equation (34), then, in accordance with Proposition 4.1, to obtain the explicit general solution to (33) we only need to get a particular solution to (33).

First of all we will obtain the general solution to the simpler equation

$$y^{(\alpha}-\lambda y=f(x) \qquad (x>a) \qquad (45)$$

where $y^{(\alpha}=D_{a+}^{\alpha}\ y$.

Proposition 4.1 . Let $f\in L_1(a,b)\cap C[(a,b)]$. Then equation (45) admits

$$y_g = ce_\alpha^{\lambda(x-a)} + y_p \ , \qquad (46)$$

as a general solution in which

$$y_p = e_\alpha^{\lambda x} *^a \ f(x) \ , \qquad (47)$$

is a particular solution to (45), with $*^a$ being the following convolution

$$g(x) *^a f(x) = \int_a^x g(x-t)f(t)dt. \qquad (48)$$

In addition, $y_p(a+) = 0$, if $f(x) \in C([a,b])$ and $\left(I_{a+}^{1-\alpha} \, y_p\right) \ (a+) = 0$, if $f(x) \in C_{1-\alpha}([a,b])$.

Theorem 4.2. A particular solution to equation (33) is given by

$$y_p = G_\alpha(x) \ *^a \ f(x) \qquad (49)$$

where $G_\alpha(x)$ is

$$G_\alpha(x) = \prod_{j=1}^{k} *^a \left(\prod_{l=1}^{\sigma_j} *^a \ e_\alpha^{\lambda(x-a)} \right) \qquad (50)$$

where $\{\lambda_j\}_{j=1}^{k}$ are the k distinct complex roots of the characteristic polynomial (36) with multiplicity $\{\sigma_j\}_{j=1}^{k}$, respectively.

In addition, $y_p(a+) = 0$ if $f(x) \in C([a,b])$ and $\left(I_{a+}^{1-\alpha} \, y_p\right)(a+) = 0$ if $f(x) \in C_{1-\alpha}([a,b])$. Moreover $(I_{a+}^{1-\alpha} \, G_a)(a+) = 0$.

Remark 4.1. Since function $G_\alpha(x-\xi)$ plays the role of Green's function associated with non-homogeneous equation (33), analogous to the usual case, this function will be called Riemann-Liouville fractional Green's function.

Remark 4.2. Analogous results can be obtained if we consider the Caputo fractional derivative (31) or (32) instead of the Riemann-Liouville fractional derivative, by using the Mittag-Leffler function

$$E_\alpha\big(\lambda(x-a)\big) = \sum_{k=0}^{\infty} \frac{\lambda^k (x-a)^{k\alpha}}{\Gamma(\alpha k+1)} \qquad (\alpha > 0) \qquad (51)$$

instead of the α-exponential function $e_\alpha^{\lambda(x-a)}$.

Example 4.2. Let us consider the equation

$^{C}D_{a+}^{2\alpha}\, y + \lambda^2 y = 0.$ (52)

Its corresponding characteristic polynomial is $P_2(x) = x^2 + \lambda^2$ and so the fundamental set of solutions to (52) is

$\{\cos_\alpha^*[\lambda(x-a)],\ \sin_\alpha^*[\lambda(x-a)]\}$ (53)

where

$\cos_\alpha^*[\lambda(x-a)] = Re\{E_\alpha(\lambda(x-a)^\alpha)\}$ (54)

and

$\sin_\alpha^*[\lambda(x-a)] = Im\{E_\alpha(\lambda(x-a)^\alpha)\}$ (55)

We point out here that the $\sin_\alpha^*(x)$ and $\cos_\alpha^*(x)$ functions are a new generalization of the usual $\cos(x)$ and $sin(x)$ functions , which, like the $sin_\alpha(x)$ and $\cos_\alpha(x)$ functions, could play a fundamental role , for instance, in the development of a fracti- onal Fourier theory, or of Weierstrass type fractal functions , which are solutions to elementary fractional differential equations.

In addition, the results previously presented may be applied to RiemannLiouville **non - sequential** linear fractional differential equations . It is possible to prove the following :

Corollary 4.1. Let $f \in C_{1-\alpha}([a,b])$ and $a_0, a_1 \in R$. then equation

$D_{a+}^{2\alpha} y + a_1 D_{a+}^{\alpha} y + a_0 y = f(x) \quad (0 < \alpha \le 1)$ (56)

has the general solution

$y(x) = C_1 z_1(x) + C_2 z_2(x) + z_p(x) - \frac{C}{\Gamma(\alpha)}(x-a)^{\alpha-1}$ (57)

where $z_i\ (i = 1,2)$ is a fundamental system of solutions to the homogeneous sequential fractional differential equation

$D_{a+}^{2\alpha}\, z + a_1 D_{a+}^{\alpha}\, z + a_0\, z = 0$, (58)

and

$z_p(x) = z_1(x) *^a z_2(x) *^a [f(x) + a_0 C(x-a)^{\alpha-1}]$ (59)

is a particular solution to the non-homogeneous equation

$D_{a+}^{2\alpha}\, z + a_1 D_{a+}^{\alpha}\, z + a_0\, z = f(x) + a_0 C(x-a)^{\alpha-1}$ (60)

where $C,\ C_1$ and C_2 are real constants such that $C_1 + C_2 = C$ if the roots of the characteristic equation of (58) are different, or $C_1 = C$, if they are not .

Example 4.3. Let $0 < \alpha \leq 1$ and $f \in C_{1-\alpha}([a, b])$. A general solution to equation

$$D_{a+}^{2\alpha} y - 2D_{a+}^{\alpha} y + y = f(x) \quad (x > a), \tag{61}$$

is

$$y_g(x) = Ce_{\alpha}^{(x-a)} + C_2\, \varepsilon_{\alpha,1}^{(x-a)} + u(x) - \frac{C}{\Gamma(\alpha)}(x-a)^{\alpha-1} \tag{62}$$

C_2 and C being two arbitrary real constants, and

$$u(x) = e_{\alpha}^{(x-a)} *^{\alpha} \varepsilon_{\alpha,1}^{(x-a)} *^{\alpha} \left[f(x) + \frac{C}{\Gamma(\alpha)}(x-a)^{\alpha-1}\right], \tag{63}$$

Example 4.4. The ordinary differential equation

$$x^{`}(t) - a^2 x(t) = 0, \tag{64}$$

according to the relation given in (11), may be transformed into the sequential linear fractional differential equation

$$(D_{0+}^{2\alpha} x)(t) - a^2 x(t) = 0 \ \left(\alpha = \frac{1}{2}\right), \tag{65}$$

whose general solution is

$$x(t) = C_1 e_{\alpha}^{at} + C_2 e_{\alpha}^{-at}. \tag{66}$$

Any solution to (64) is included in the family of solutions to (66) because $x(0) < \infty$ and so $C_2 = -C_1$. Then

$$x(t) = C_1 \sum_{j=1}^{\infty} \frac{[1-(-1)^j]\alpha^j\, t^{j\alpha+\alpha-1}}{\Gamma[(j+1)\alpha]}. \tag{67}$$

which is the well-known general solution to (64).

However, $x(t) = e_{\alpha}^{at}$ is a solution to (65) but it is not a solution to (64).

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