Upward Closed Topology On $\mathbb{N}$ With Its Effect On The Levels Of $\mathbb{N}$

Salahddeen Khalifa
Department of Mathematics
Faculty of science
University of Gharyan
Gharyan Libya
e-mail: Ksalahddeen@yahoo.com

December 31, 2022

Abstract

To find a connection between the usual divisibility relation that is defined on the set of natural numbers $\mathbb{N}$ and topological concepts on $\mathbb{N}$ for some topologies that are defined on $\mathbb{N}$, we have defined topology on $\mathbb{N}$ depending on the usual divisibility relation that is defined on $\mathbb{N}$, this topology contains the set of all upward closed subsets of $\mathbb{N}$ and it is called upward closed topology on $\mathbb{N}$. In the beginning we established some roles about upward closed subsets of $\mathbb{N}$. We have proved the relationship between the usual divisibility relation that is defined on $\mathbb{N}$ and the limit points. We concluded that the topological relation between the levels of $\mathbb{N}$ is the numbers that are in the lower levels are limit points to the up levels.

1 Introduction

(a) Numbers Concepts

If we have the set of natural numbers $\mathbb{N}$, $a, b \in \mathbb{N}$ we say that a divides b (written $a|b$) if there is a natural number c such that $b = ac$. $c \in \mathbb{N}$ is the greatest common divisor of a and b (written $c = (a, b)$) if and only if $c|a, c|b$ and if $d|a$ and $d|b$ then $d \leq c$. A prime number is a natural number greater than 1 and has no divisors other than 1 and itself, and we denote the set of prime numbers by P. If $a, b \in \mathbb{N}, (a, b) = 1$ we say that a and b are relatively prime.

Theorem 1.1 ***The Unique Factorization Theorem*** *[2]*
Any natural number greater than one can be written as a product of primes in one and only one way.

i.e for any $n \in \mathbb{N}, n > 1$ *can be written in exactly one way in the form* $n = p_1^{e_1} p_2^{e_2} .....p_k^{e_k}$ *,where* $e_i \geq 0, i = 1, 2, ..., k,$ $p_i \in P$ *, and* $p_i \neq p_j$*. We call this representation by the* ***prime-power decomposition of n***

As a result of the Unique Factorization Theorem,the set of natural numbers can be divided in to infinitely many levels $L_i, i \geq 0$ such that $L_0 = \{1\}$,
$L_i = \{a_1 a_2 ....a_i : a_1, a_2, ..., a_i \in P\}, i \geq 1,$ $\mathbb{N} = \bigcup\limits_{i=1}^{\infty} L_i,$ $L_i \cap L_j = \emptyset, i \neq j$
$(\bigcap\limits_{i=0}^{\infty} L_i = \emptyset)$

(b) Topological Concepts

A topology on a set X is a collection τ of subsets of X called the open sets satisfying the following: the empty set $\emptyset$ and X belong to τ, any union of elements of τ belongs to τ, and any finite intersection of elements of τ belongs to τ. We say that (X, τ) is a topological space. A base for a topological space (X, τ) is a collection $\text{ß} \subset \tau$ such that $\tau = \{ \bigcup\limits_{B \in \eta} B : \eta \subset \text{ß}\}$.
A subset F of X is closed if $X - F \in \tau$. A limit point of a set A in a topological space X is a point $x \in X$ such that each open set of x contains some points of A other than x. i.e $A \cap (U - \{x\}) \neq \emptyset$, for any open set $U, x \in U$, we denote the set of limit points to A by A'. A set A is closed if and only if contains all of its limit points. Closure of a set A in a topological space X is denoted by $\overline{A}$, and defined by $\overline{A} = A \cup A'$. A set A in a topological space X is dense if $\overline{A} = X$.

2 Upward Closed Subsets Of $\mathbb{N}$

If A is an infinite subset of $\mathbb{N}$, and if the set that contains all the numbers that are divided by some numbers in A is equal to the set A, in this case, the set A is called an upward closed subset of $\mathbb{N}$ and the set that contains all upward closed subsets of $\mathbb{N}$ is called the collection of the upward closed subsets of $\mathbb{N}$ as it will be explained in the following definition.

Definition 2.1 *[1] (a) For any* $a \in \mathbb{N}$, $a \uparrow = \{n \in \mathbb{N} : a|n\} = \{ma : m \in \mathbb{N}\}$
(b) The collection of upward closed subset of $\mathbb{N}$ *is* $\mu = \{A \subseteq \mathbb{N} : A = A \uparrow\}$ *where* $A \uparrow = \{n \in \mathbb{N} : \exists a \in A, a|n\}$

Example 2.1 $1 \uparrow = \mathbb{N}$,$2 \uparrow = \{2, 4, 6, ...\}$, $P \uparrow = \mathbb{N} - \{1\}$,
$(\mathbb{N} - \{1\}) \uparrow = \mathbb{N} - \{1\}$.

Lemma 2.1 *(a)* $\emptyset, \mathbb{N} \in \mu$, $L_i \notin \mu$ *for all* $i \geq 0$.
(b) If $A \in \mu$*, then* $A = \emptyset$ *or infinite set.*

Proof : (a) It is obvious that $\mathbb{N} \in \mu$. If $\emptyset \uparrow \neq \emptyset$ then there exist $n \in \mathbb{N}, n \in \emptyset \uparrow$ and $a \in \emptyset$ such that $a|n$, so we have a contradiction.Thus $\emptyset \uparrow = \emptyset$.
$L_i \uparrow = \bigcup\limits_{j=i}^{\infty} L_j$ for all $i \geq 0$.
(b) Let $A \in \mu$. If we suppose that $A \neq \emptyset, A$ is finite set, let $A = \{a_1, a_2, ..., a_n\}$,

where $a_1 < a_2 < ... < a_n$ and let $b \in \mathbb{N}, b = ca_n$, where $c > 1$, then $b \in A \uparrow = A$, $b \notin A$, so we have a contradiction. Thus $A = \emptyset$ or A is an infinite set. ■

Lemma 2.2 *(a) $n \uparrow, A \uparrow$ are upward closed for any $n \in \mathbb{N}, A \subseteq \mathbb{N}$.*
(b) If A is upward closed then $A \uparrow = (A \uparrow) \uparrow$.

Proof : (a) Obvious.

(b) Let $A \in \mu$,

$$\begin{aligned}(A \uparrow) \uparrow &= \{n \in \mathbb{N} : \exists a \in A \uparrow, a|n\} \\ &= \{n \in \mathbb{N} : \exists a \in A, a|n\} = A \uparrow . \blacksquare\end{aligned}$$

Lemma 2.3 *(a) If $n_1, n_2 \in \mathbb{N}, n_1|n_2$, then for any $A \in \mu$ contains n_1 contains also n_2.*
(b) If $n_1, n_2 \in \mathbb{N}, n_1|n_2$ then $n_2 \uparrow \subseteq n_1 \uparrow$.

Proof : (a) Let $n_1, n_2 \in \mathbb{N}, n_1|n_2$, and let $A \in \mu, n_1 \in A$, so there exists $a \in A, a|n_1$, so $a|n_2$. Thus $n_2 \in A$.

(b) Let $n_1, n_2 \in \mathbb{N}, n_1|n_2, n \in n_2 \uparrow$, so $n = m_1 n_2, m_1 \in \mathbb{N}$, and $n = (m_1 m_2) n_1$, $m_2 \in \mathbb{N}$,so $n \in n_1 \uparrow$. Thus $n_2 \uparrow \subseteq n_1 \uparrow$. ■

Lemma 2.4 *(a) If $A \subseteq B$, then $A \uparrow \subseteq B \uparrow$ for any $A, B \subseteq \mathbb{N}$.*
(b) If $A, B \in \mu$, then $A \cap B \in \mu$, $(\bigcap_{i=1}^{n} A_i \in \mu$, where $A_i \in \mu, i = 1, 2, ..., n)$.
(c) If $A, B \in \mu$, then $A \cup B \in \mu$, $(\bigcup_{i=1}^{n} A_i \in \mu$, where $A_i \in \mu, i = 1, 2, ..., n)$.
(d) $\bigcup_{i=1}^{\infty} A_i \in \mu$, where $A_i \in \mu, i = 1, 2, ...$

Proof : (a) If $A, B \subseteq \mathbb{N}, A \subseteq B, n \in \mathbb{N}, n \in A \uparrow$, so there exists $a \in A, a|n$, so $a \in B, n \in B \uparrow$. Thus $A \uparrow \subseteq B \uparrow$.

(b) Let $A, B \in \mu$

$$\begin{aligned}(A \cap B) \uparrow &= \{n \in \mathbb{N} : \exists a \in A \cap B, a|n\} \\ &= \{n \in \mathbb{N} : \exists a \in A, a|n\} \cap \{n \in N : \exists a \in B, a|n\} \\ &= A \uparrow \cap B \uparrow = A \cap B\end{aligned}$$

(c) Let $A, B \in \mu$

$$\begin{aligned}(A \cup B) \uparrow &= \{n \in \mathbb{N} : \exists a \in A \cup B, a|n\} \\ &= \{n \in \mathbb{N} : \exists a \in A, a|n\} \cup \{n \in \mathbb{N} : \exists a \in B, a|n\} \\ &= A \uparrow \cup B \uparrow = A \cup B\end{aligned}$$

(d) similar to (c) ■

Theorem 2.1 *(a) If $A \subseteq \mathbb{N}$ is upward closed, then $A = \bigcup_{a\in A} a\uparrow$.*
(b) $\mu = \{\bigcup_{a\in A} a\uparrow : A \in \mu\}$.

Proof : (a) Let $A \subseteq N$ is upward closed, and $n \in A$, then there exists $a \in A$ such that $a|n$, so $n \in a\uparrow$, and $n \in \bigcup_{a\in A} a\uparrow$. Thus $A \subseteq \bigcup_{a\in A} a\uparrow$.
On the other hand, if $n \in \bigcup_{a\in A} a\uparrow$, then there exists $a \in A, n \in a\uparrow$, so $a|n$, and $n \in A\uparrow$. Thus $\bigcup_{a\in A} a\uparrow \subseteq A\uparrow = A$. Therefore $A = \bigcup_{a\in A} a\uparrow$.
(b) By (a) and definition of upward closed. ∎

Now, since the numbers that are in the lower levels can't be divided by the numbers that are in the up levels, so when we take off the lower levels from $\mathbb{N}$ we will get upward closed sets.

Theorem 2.2 $\bigcup_{i=k}^{\infty} L_i, k = 0, 1, 2, ...$ *are upward closed subsets of* $\mathbb{N}$.

Proof : If $k = 0$, then $\bigcup_{i=0}^{\infty} L_i = \mathbb{N}$ is upward closed.
If $k = 1$, since 1 can't be divided by any number in $\bigcup_{i=1}^{\infty} L_i$, so $(\bigcup_{i=1}^{\infty} L_i)\uparrow = \bigcup_{i=1}^{\infty} L_i$
If $k = 2, 3, ...$, 1 can't be divided by any number in $\bigcup_{i=2}^{\infty} L_i$. Let $n \in L_j$, where $1 \leq j < k$, then $n = {a_1}^{n_1}{a_2}^{n_2}..{a_j}^{n_j}$, where $n_1 + n_2 + .. + n_j = j$, and $a_1, a_2, ...a_j \in P$. n can't be divided by any number in L_i, where $i \geq k$. Thus $(\bigcup_{i=k}^{\infty} L_i)\uparrow = \bigcup_{i=k}^{\infty} L_i \ \forall\ k = 0, 1, 2, ...$ ∎

3 Upward Closed Topology On $\mathbb{N}$

A collection of upward closed subsets of $\mathbb{N}$ defined topology on $\mathbb{N}$, with this topology there is a connection between the usual divisibility that is defined on $\mathbb{N}$ and the limit points. And since the levels of natural numbers have been established with divisibility, then we will look for the topological relation between all the levels $L_i, i \geq 0$ with this topology.

Lemma 3.1 *(a) The collection of upward closed subsets of N is defined topology on $\mathbb{N}$.*
(b) The collection $\beta = \{n\uparrow : n \in \mathbb{N}\}$ is a basis for μ.

Proof (a) Let $\mu = \{A \subseteq \mathbb{N} : A = A\uparrow\}$ where $A\uparrow = \{n \in \mathbb{N} : \exists a \in A, a|n\}$.
(1) Since $N\uparrow = \mathbb{N}, \emptyset\uparrow = \emptyset$, then $\mathbb{N}, \emptyset \in \mu$.
(2) If $A_1, A_2, ..., A_n \in \mu$, then by (Lemma(2.4)(b)) $\bigcap_{i=1}^{n} A_i \in \mu$
(3) Let $A_\gamma \in \mu, \gamma \in \Gamma$, then $A_\gamma\uparrow = A_\gamma \ \forall\ \gamma \in \Gamma$ and

$$(\bigcup_{\gamma\in\Gamma} A_\gamma)\uparrow = \{n \in \mathbb{N} : \exists a \in \bigcup_{\gamma\in\Gamma} A_\gamma, a|n\}$$
$$= \bigcup_{\gamma\in\Gamma} \{n \in \mathbb{N} : \exists a \in A_\gamma, a|n\}$$
$$= \bigcup_{\gamma\in\Gamma} (A_\gamma \uparrow) = \bigcup_{\gamma\in\Gamma} A_\gamma$$
.

So $\bigcup_{\gamma\in\Gamma} A_\gamma \in \mu$.
Thus μ defined topology on $\mathbb{N}$.

(b) By (Theorem (2.1)(a)) for any $A \in \mu, A = \bigcup_{n\in A} n\uparrow$.
Thus ß is a basis for μ. ■

We denote to a topological space $\mathbb{N}$ with μ by $(\mathbb{N}, \mu)$

Lemma 3.2 *In the space* $(\mathbb{N}, \mu)$ *with upward closed topology.*
(a) a is a limit point for $\{b\}$ if and only if $a|b, a \neq b$.
(b) $\{a\}' = \{n \in \mathbb{N} : n|a, n \neq a\}$, for any $a \in \mathbb{N}$.
(c) n is a limit point for A if and only if there exists $a \in A, n|a, n \neq a$.
(d) $A' = \{n \in \mathbb{N} : \exists a \in A, n|a, n \neq a\}$ for any $A \subseteq \mathbb{N}$.
(e) If $a|b$, then $\{a\}' \subset \{b\}'$.

Proof : (a) ($\Rightarrow$) Let $a, b \in \mathbb{N}, a$ is a limit point of $\{b\}$, since $a\uparrow$ is an open set, $a \in a\uparrow$. So $\{b\} \cap (a\uparrow -\{a\}) \neq \emptyset$, and $b \in a\uparrow$. Thus $a|b, a \neq b$.
($\Leftarrow$) Let $a, b \in \mathbb{N}, a|b, a \neq b$, and let U is an open set, $a \in U$, by (Lemma (2.3)(a)) we have $b \in U$, so $\{b\} \cap (U - \{a\}) \neq \emptyset$. Thus a is a limit point for $\{b\}$.

(b) By (a) if $n \in \mathbb{N}, n|a, n \neq a$, then n is a limit point for $\{a\}$.
Thus $\{a\}' = \{n \in \mathbb{N}, n|a, n \neq a\}$.

(c) ($\Rightarrow$) Let $n \in \mathbb{N}, A \subseteq \mathbb{N}, n$ is a limit point for A, since $n\uparrow$ is open set, $n \in n\uparrow, soA \cap (n\uparrow -\{n\}) \neq \emptyset$, and there exists $a \in A, n\uparrow, n|a, n \neq b$.
($\Leftarrow$) Let $A \subseteq \mathbb{N}, a \in A, n \in \mathbb{N}, n|a, n \neq a$, and let U is an open set, $n \in U$. By (Lemma(2.3)(a)) we have $a \in U$, so $A \cap (U - \{n\}) \neq \emptyset$. Thus n is a limit point for A.

(d) By (c) if $a \in A, n|a, n \neq a$, then n is a limit point for A.
Thus $A' = \{n \in \mathbb{N} : \exists a \in A, n|a, n \neq a\}$.

(e) Let $a|b$, and $n \in \mathbb{N}, n \in \{a\}'$, then by (a) $n|a$, so $n|b$, and by (a) $n \in \{b\}'$.
Thus $\{a\}' \subset \{b\}'$. ■

Corollary 3.1 *In the space* $(\mathbb{N}, \mu)$*:* $(a, b) = 1$ *if and only if* $\{a\}' \cap \{b\}' = \{1\}$.

Proof : ($\Rightarrow$) let $(a,b)=1$, and $n\in\{a\}'\cap\{b\}'$. By (Lemma (3.2)(a)) $n|a, n|b$. So $n=1$, and $\{a\}'\cap\{b\}'=\{1\}$.
($\Leftarrow$) Let $\{a\}'\cap\{b\}'=\{1\}$, then by (Lemma(3.2)(b))
$\{n\in\mathbb{N}:n|a\}\cap\{n\in\mathbb{N}:n|b\}=\{1\}$. Thus $(a,b)=1$. ■

Corollary 3.2 *In the space* $(\mathbb{N},\mu)$. $A\subseteq\mathbb{N}$ *is closed if and only if for any* $a\in A, b|a$. $b\in A$.

Proof : ($\Rightarrow$) Let $A\subseteq\mathbb{N}$ is closed, and $a\in A, b|a$. So $b\in A'$, and since the closed set contains all its limit points, so $b\in A$.
($\Leftarrow$) Let $A\subseteq\mathbb{N}, a\in A, b|a, b\in A$. So $b\in A'$ and since b is an arbitrary number in $\mathbb{N}$, so A contains all its limit points and is closed. ■

Corollary 3.3 *In the space* $(\mathbb{N},\mu)$. $n\uparrow'=\mathbb{N}$ *for any* $n\in\mathbb{N}$.

Theorem 3.1 *In the space* $\mathbb{N}$ *with the upward closed topology, the numbers that are in the lower levels are limit points to the up levels.*
i.e $L_i'=\bigcup_{j=0}^{i-1}L_j, i=0,1,2,....$

Proof If $i=0$, then for any $n\in\mathbb{N}, n\uparrow$ is an open set and $\{1\}\cap(n\uparrow-\{n\})=\emptyset$, so $n\notin L_0'$.Thus $L_0'=\emptyset$
If $i=1$, since $1\neq n$ for any $n\in L_1, 1|n$, so by (Lemma (3.2)(c)) $1\in L_1'$
For any $a\in L_i, i=1,2,3,....$ since $a\uparrow$ is an open, and $L_1\cap(a\uparrow-\{a\})=\emptyset$, so $a\notin L_1'$. Thus $L_1'=\{1\}$
If $i=2,3,.......$, since 1 divided and doesn't equal any number in $L_i, i\geq 2$ so by (Lemma(3.2)(c)) $1\in L_i', i=2,3,.....$
For any two levels L_i, L_j, $j<i, i=2,3,...., j=1,2,....$, let $a\in L_i, U$ is an open set,$a\in U$, $a=n_1^{e_1}n_2^{e_2}.....n_j^{e_j}, n_1,n_2,....,n_j\in P, e_1+e_2+....+e_j=j$
$n=n_1^{e_1}n_2^{e_2}.....n_j^{e_j}n_{j+1}^{e_{j+1}}....n_i^{e_i}\in L_i, n_1,n_2,....,n_i\in P, e_1+e_2+....+e_i=i$. So $a|n$ and by (Lemma (3.2)(c)) $a\in L_i', i=2,3,....$
If $a\in L_k, k\geq i$, then $a\uparrow$ is an open set $a\in a\uparrow, L_i\cap(a\uparrow-\{a\})=\emptyset$, so a isn't a limit point to L_i. Thus $L_i'=\bigcup_{j=0}^{i-1}L_j$ for all $i=2,3,.....$

Therefore $L_i'=\bigcup_{j=0}^{i-1}L_j$ for all $i=0,1,2,.....$ ■

Corollary 3.4 *In the space* $(\mathbb{N},\mu)$
(a) $(\bigcup_{i=k}^{\infty}L_i)'=\mathbb{N}$ *for all* $k=1,2,......$
(b) $\overline{(\bigcup_{i=k}^{\infty}L_i)}=\mathbb{N}$ ($\bigcup_{i=k}^{\infty}L_i$ *are dense in* $\mathbb{N}$ *for all k=1,2 ....)*

Proof (a) Since for any $n\in\mathbb{N}$ there exists $a\in\bigcup_{i=k}^{\infty}L_i$ for all $k=1,2,....$ such that $n|a, n\neq a$, then by (Lemma (3.2)(c)) n is a limit point to $\bigcup_{i=k}^{\infty}L_i$. Thus

$(\bigcup_{i=k}^{\infty} L_i)' = \mathbb{N}$ for all $k = 1, 2, ...$

(b) Since $(\bigcup_{i=k}^{\infty} L_i)' \subseteq \overline{(\bigcup_{i=k}^{\infty} L_i)}$, then $\overline{(\bigcup_{i=k}^{\infty} L_i)} = \mathbb{N}$ for all $k = 1, 2, ...$ ∎

References

[1] *B.Sobot, Divisibility order in $\beta\mathbb{N}$ arXiv:1511.01731v2 [math.LO] 17 May 2016*

[2] *D.Underwood, Elementary number Theory.Second edition 2008, Dover publications,Inc.Mineola, New York*

[3] *L.Azriel, Basic set Theory 2002, Dover publications, DJVU, 5.13 MB*

[4] *S.Khalifa, "Divisibility in the Stone-Cech Compactification of $\mathbb{N}$" (2018)*

[5] *S.Khalifa, A.Abominjil, Levels of Natural Numbers 2022, Gharyan University Journal*

[6] *S.Willard, General Topology. 2004 Dover Publications,Inc,Mineola,New York, o-486-43479-6*