الرسوم البيانية المرتبطة بالزمر التبديلية المنتهية

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الملـخـص:
بفرض أن A زمرة منتهية، الرسم البياني الغير موجه $G(A)$ للزمرة المنتهية A عرف على أنه البيان البسيط الغير موجه الذي مجموعة حوافه هي $G(A)$ و الرأســين المختلفــين u, v يكونان مرتبطين إذا وفقط إذا كان $u^2 \neq v^2$. في هذا البحث لأي زمرة منتهية $A = (\mathbb{Z}_n, .)$ سوف نقوم بدراسة : درجة الرؤوس، عدد الحواف للرسم البياني $G(A)$، إختلاف المراكز، قطر الرسم البياني، حجم الرسم البياني، الأعداد اللونية للرؤوس، الأعداد اللونية للحواف، بالإضافة إلى ذلك، إثبات وجود الدوائر الهاملتينونية والدوائر الأولريانية.

Graphs Related to Finite Abelian Groups

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Abstract: Let A be a finite abelian group, An undirected graph $G(A)$ of a finite group A is introduced and defined as an undirected simple graph whose vertex set is $V(\text{G})$ and two distinct vertices u and v are adjacent if and only if $u^2 \neq v^2$. In this paper, for a finite group $A = (\mathbb{Z}_n, .)$, the degrees of vertices, the number of edges of $G(A)$, the eccentricity, diameter, girth, chromatic number are computed. Furthermore, Hamiltonian and Eulerian cycles of G are proposed.

Keywords: *Abelian Group, Hamiltonian Cycle, Degree, Simple Graph, Chromatic Number, Eccentricity, Diameter, Girth.*

1. Introduction

The study of graphs of abelian groups was introduced and widely researched since many group properties can be represented by graphs. In particular, the study of graphs of the group $\mathbb{Z}_n$ of integers modulo n reveals interesting relations between group theory, number theory and graph theory; algebraic tools help to understand graphs properties and vise versa. Given an algebraic structure A, there are different formulations to associate a directed or undirected graph to A, and the algebraic properties of A are studied in terms of properties of associated graphs.

There are many papers on assigning a graph to a group and algebraic properties of the group by using the associated graph; for instance, see [1–4]. In recent years, there has been growing interest in the graphs associated with the finite group $\mathbb{Z}_n$ of integers modulo n, see [5,6]. The idea of studying the interplay between group-theoretic properties of a group A and graph-theoretic properties of a graph defined after it is quite recent.

Given a abelian group $\mathbb{Z}_n$, we identify an undirected graph $G(\mathbb{Z}_n)$ of a finite abelian group $\mathbb{Z}_n$ as an undirected graph whose vertex set is $V(G) = \mathbb{Z}_n$ and two distinct vertices u and v are adjacent if and only if $u^2 \neq v^2$. By this adjacency, all elements of a group are represented in a connected graph $G = (V, E)$, where V is the set of all vertices and E is the set of all edges.

Given a graph $G = (V, E)$, we study a graph degree of a vertex u, denoted by $deg(u)$. Also, we are interested in the eccentricity of a graph vertex v, denoted by $ecc(v)$, the diameter, denoted by $diam(G)$, and the girth of G, denoted by $g\mathrm{r}(G)$. A

Hamiltonian cycle and Hamiltonian graph are also in our concentration, and of course, by using graph coloring we have found the vertex chromatic number $\chi\ (G)$ and the edge chromatic number $\chi'(G)$ as well.

2. Preliminary

In this section we review some primary concepts from graph theory, and we refer to [7, 8] for the notions are used.

Two vertices are said to be *adjacent vertices* if there is an edge (arc) connecting them.

A simple graph is a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex. In other words, a simple graph is a graph without loops and multiple edges.

Adjacent edges are edges that share a common vertex. The *degree of a vertex* is the number of edges incident with that vertex. A graph in which every vertex has the same degree is called a *regular graph*.

A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

A path is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. A path that does not repeat vertices is called a *simple path*. *An Euler path* is a path that travels through all edges of a connected graph. *An Euler circuit* is a circuit that visits all edges of a connected graph. *An Eulerian cycle* in an undirected graph is an Eulerian circuit that uses each edge exactly once.

A circuit is a path that begins and ends at the same vertex. A circuit that doesn't repeat vertices is called *a cycle*.

A graph is said to be *connected* if any two of its vertices are joined by a path. A graph that is not connected is a *disconnected* graph.

The *girth* of an undirected graph is the length of the shortest cycle contained in the graph. If the graph does not contain any cycles, its girth is defined to be infinity.

The *eccentricity* of a graph vertex v in a connected graph G, denoted by $ecc(v)$, is the maximum graph distance between v and any other vertex u of G. For a disconnected graph, all vertices are defined to have infinite eccentricity.

Graph center is the set of vertices with minimum eccentricity in the graph G.

The *diameter* of a graph is the maximum eccentricity of any vertex in the graph. That is, it is the greatest distance between any pair of vertices. To find the diameter of a graph, first, find the shortest path between each pair of vertices. The greatest length of any of these paths is the diameter of the graph.

The radius of a graph is the minimum graph eccentricity of any graph vertex in a graph. A disconnected graph, therefore, has an infinite radius.

Graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices are of the same color; this is called a vertex coloring.

An edge coloring or line coloring of a graph G is an assignment of colors to its edges (lines) so that no two adjacent edges (lines) are assigned the same color. An n-edge coloring of G is an edge coloring of G which uses exactly n colors.

The chromatic number, $\chi(G)$, of a graph G is the smallest number of colors for $V(G)$ so that adjacent vertices are colored differently.

A clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent. That is, a clique of a graph G is an induced subgraph of G that is complete. Moreover, the clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.

3. Main Results

In this section, we investigate a graph degree of a vertex u, a diameter, the girth, a Hamiltonian cycle and Hamiltonian graph, chromatic number $\chi\ (G)$ and the edge chromatic number $\chi'(G)$ of the graph G.

One observes for any prime integer $n < \infty$, $G(\mathbb{Z}_n)$ doesn't contain loops, which means that it is a simple graph.

Suppose that $(\mathbb{Z}_n,\ \cdot)$ is a group where n is a prime, then every two vertices $u = i, v = n - i$ are disjoint, where $i = 1,2,3, \dots, n - 1$. For instance, in the Figure 4, the vertices $u = 6$, $v = 7$ are clearly disjoint.

In the next remark, we indicate the adjacency of vertex $v = 1$ for some prime integer $n \geq 5$.

Remark 3.1. *In the graph $G(\mathbb{Z}_n)$, if $|V(G)| > 4$ then the vertex 1 must be at least adjacent at two vertices, which are: $u = 2, v = n - 2$.*

Theorem 3.1. *For every prime* $n \geq 5$, *and* $v \in G(\mathbb{Z}_n)$, *then* $deg(v) = n - 3$.

***Proof*:** Let v be any vertex in $V(G)$ then v is adjacent to all vertices in $V(G)$ except the vertex $n - v$. Since $|V(G)| = n - 1$, also v is not adjacent to itself. Thus, the number of vertices that are adjacent to v are $n - 1 - 2 = n - 3$.

Theorem 3.2. *The eccentricity of a vertex* $\boldsymbol{v}$ *in* $\boldsymbol{G}(\mathbb{Z}_{\boldsymbol{n}})$ *is 2***.**

***Proof*:** Let v be an element in $V(G)$ such that $v \neq n - 1$**.** Since $v, n - v$ are disjoint (by definition) that means $ecc(G)$ can not be 1. However, the vertex $v + 1$ is adjacent to both v and $n - v$ (easy to prove), which is the maximum distance between v and any other vertex u in $G(\mathbb{Z}_{\boldsymbol{n}})$**.** Therefore, the $ecc(v)$ must be 2.

Corollary 3.1: *The diameter of* $G(\mathbb{Z}_n)$ *is 2.*

Proof: By the definition of the diameter and theorem 3.2, the proof follows.

Corollary 3.2: *The radius* $r(G)$ *of the graph* $G(\mathbb{Z}_n)$ *is 2.*

Proof: By the definition of radius and theorem 3.2, the proof follows.

Proposition 3.1: *The graph center of* $G(\mathbb{Z}_n)$ *is* $\{1, 2, \dots, n - 1\}$.

Proof: Since the eccentricity of all vertices in $G(\mathbb{Z}_n)$ is 2 [Corollary 3.2]. Thus, the central set of $G(\mathbb{Z}_n)$ is all vertices in $V(G)$.

Theorem 3.3. *If* n *be any prime number. Then, the girth of the graph* $G(\mathbb{Z}_n)$ *is* $gr(G) = \begin{cases} \infty & where\ n < 5 \\ 4 & where\ n = 5 \\ 3 & where\ n \geq 7 \end{cases}$.

***Proof*:**

1- If $n < 5$ then either $n = 2$, that is $|V(G)| = 1$, so we have a single vertex without any edge; or $n = 3$ that is $|V(G)| =$

2, so we have only two vertices also without any cycle. Therefore, $gr(G) = \infty$.

2- If $n = 5$ then $G(\mathbb{Z}_n)$ the vertices 1, 4 are not adjacent and 2, 3 also are not adjacent. Therefore, $e_{12}e_{24}e_{43}e_{31}$ makes a cycle of length 4. Hence $gr(G) = 4$. See Figure 1.

3- If $n \geq 7$ then by theorem 3.1 tow vertices $1, 2$ are adjacent to the vertex 3 and hence we must have (at least) one smallest cycle $e_{13}e_{32}e_{21}$ which has length 3. Hence, $gr(G) = 4$.

Theorem 3.4: *The graph $G(\mathbb{Z}_n)$ is Hamiltonian.*

Proof: Let n be a prime number such that $n > 4$, and $V(G) = \{1, 2, \dots, n-1\}$. Now if we divided V in to two sets V_1, V_2 such that: $V_1 = \{v_1 = 1, v_2 = 2, \dots, v_{\frac{n-1}{2}} = \frac{n-1}{2}\}$ and $V_2 = \{v_{\frac{n+1}{2}} = \frac{n+1}{2},\ v_{\frac{n+3}{2}} = \frac{n+3}{2},\ v_{\frac{n+5}{2}} = \frac{n+5}{2}, \dots, v_{n-1} = n-1\}$. So by theorem 3.1, all vertices in V_1 are adjacent and also all of vertices in V_2 are adjacent and by the same theorem two vertices $v_{\frac{n-1}{2}} = \frac{n-1}{2}$ and $v_{\frac{n+1}{2}} = \frac{n+1}{2}$ are disjoint, since $v_{\frac{n-1}{2}} = \frac{n-1}{2}$ is adjacent with $v_{n-1} = n-1$ and $v_{\frac{n+1}{2}} = \frac{n+1}{2}$ is adjacent with $v_1 = 1$ then we can define a cycle C as follows:

$$v_1 = 1 \leftrightarrow v_2 = 2 \leftrightarrow \cdots \leftrightarrow v_{\frac{n-1}{2}} \leftrightarrow v_{n-1} \leftrightarrow v_{n-2} \leftrightarrow v_{n-3} \leftrightarrow \cdots \leftrightarrow v_{\frac{n+1}{2}} \leftrightarrow v_1 = 1,$$

which is a Hamilton cycle. Hence, $G(\mathbb{Z}_n)$ is a Hamiltonian graph.

Example 3.1: *In $G(\mathbb{Z}_{11})$ we can define a cycle C as:*

$$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 10 \leftrightarrow 9 \leftrightarrow 8 \leftrightarrow 7 \leftrightarrow 6 \leftrightarrow 1$$

Remark 3.2: *In every graph $G(\mathbb{Z}_n)$, where n is a prime number with $|V(G)| \geq 5$, there is another Hamiltonian undirected cycle:*

$$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow \cdots \leftrightarrow \frac{n-1}{2} \leftrightarrow \frac{n+1}{2} + 1 \leftrightarrow \frac{n+1}{2} + 2$$
$$\leftrightarrow \cdots \leftrightarrow n-1 \leftrightarrow \frac{n+1}{2} \leftrightarrow 1$$

Corollary 3.3. *The graph $G(\mathbb{Z}_n)$ is Eulerian.*

Proof: The proof follows from theorem 3.2.

Given a prime integer $n < \infty$, the subsets V_1 and V_2which are defined in the proof of the theorem 3.4 and satisfy $V = V_1 \cup V_2$, form complete subgraphs H_1, H_2 respectively. If C_1, C_2 refer to the cliques correspond to H_1, H_2 sequentially. Then, both C_1 and C_2 are maximal cliques. Therefore, the clique number of the graph G is $\omega(G) = \frac{n-1}{2}$.

Theorem 3.5. *The graph $G(\mathbb{Z}_n)$ has a chromatic number $\chi(G) = \frac{n-1}{2}$.*

Proof: For $n \geq 5$, the vertices $1, n-1$ are disjoint, so they have the same color and no other vertex can share this color, because 1 is adjacent to all other vertices. The vertices $2, n-2$ are disjoint so they have the same deferent color from 1 and no another vertex can occupy this color, because 2 is adjacent to all of other vertices, … etc., vertices $\frac{n-1}{2}$, $\frac{n+1}{2}$ are disjoint, so they have the same deferent color from all of the other previous vertices color, because $\frac{n-1}{2}$, is adjacent to all of them. Thus, $\chi(G) = \frac{n-1}{2}$.

In the graph $G(\mathbb{Z}_n)$, every vertex $v \in V$ have $n-3$ degree [Theorem 3.1], that means an edge $e \in E$ connected to v must take $n-3$ different colours, and so on. For any another vertex u we can use the same $n-3$ edges different colours with different color order, that lead us to the following corollary.

Corollary 3.4. For any prime $n \geq 5$, the edge chromatic number in graph $G(\mathbb{Z}_n)$ is $\chi'(G) = n-3$.

4. Graphs for some prime integer n

In this section we present digraphs $G(\mathbb{Z}_n)$ for some integer $n = 5, 7, 11, 13, 17, 19$. In every figure form 1-6, one may notice that the vertices $\frac{n-1}{2}, \frac{n+1}{2}$ are disjoint and their represented graphs are *(n-3)-regular* graphs.

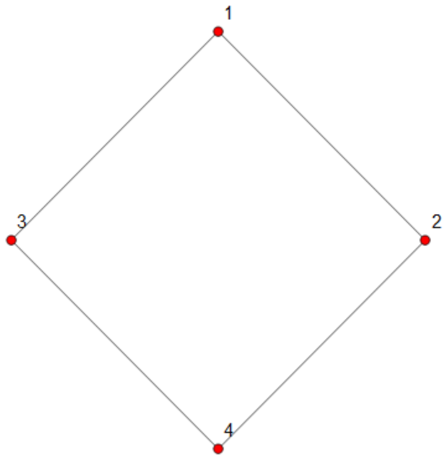

Figure 1: Shown is directed graph $G(\mathbb{Z}_5)$

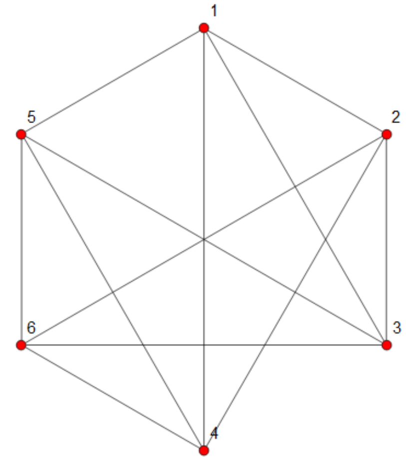

Figure 2: Shown is directed graph $G(\mathbb{Z}_7)$

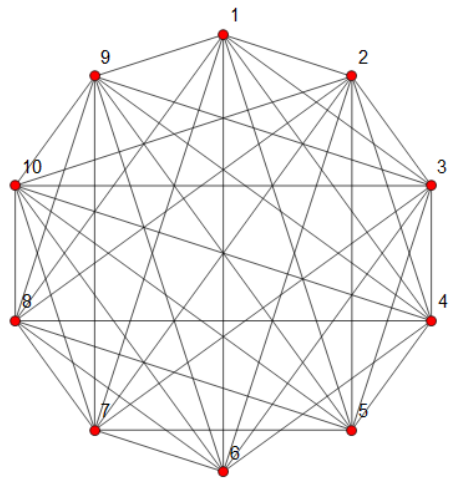

Figure 3: Shown is directed graph $G(\mathbb{Z}_{11})$

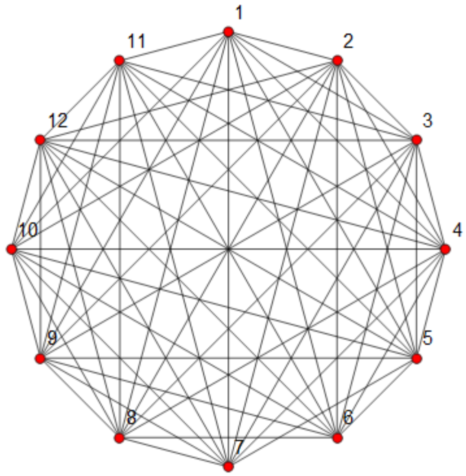

Figure 4: Shown is directed graph $G(\mathbb{Z}_{13})$

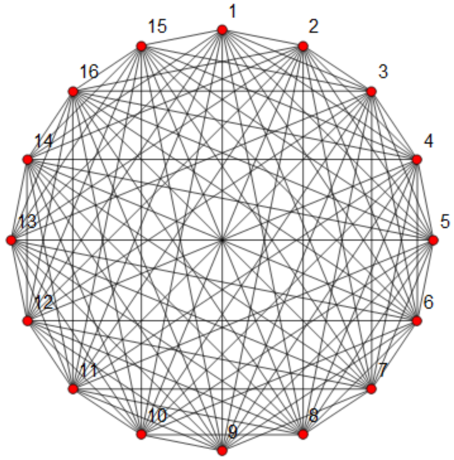

Figure 5: Shown is directed graph $G(\mathbb{Z}_{17})$

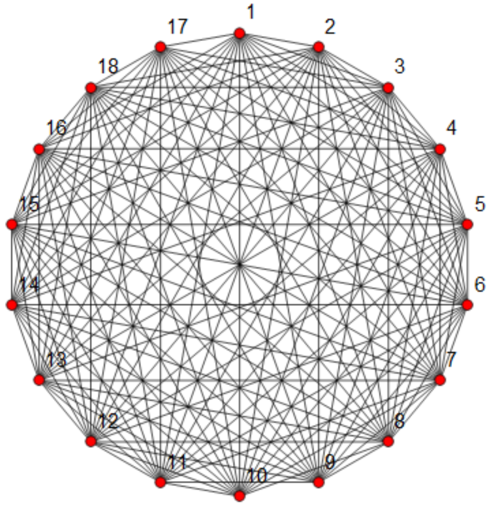

Figure 6: Shown is directed graph $G(\mathbb{Z}_{19})$

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