الحلول العددية لبعض المعادلات التفاضلية الجزئية باستخدام طريقة التحليل

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ملخص الدراسة:

في هذا البحث، تم تطبيق طريقتي Adomian decomposition method (ADM) و The modified decomposition method (MDM) ، لايجاد الحلول العددية لمعادلتي كورتيغ دي-فريز و كورتيغ دي-فريز المعدلة Korteweg –de Vries equation and modified Korteweg -de Vries equation . دقة و سرعة تقارب الحل العددي تم تأكيدها من خلال عرضه في جداول و رسوم بيانية باستخدام لغة Mathematica .

Numerical Solutions Of Some Nonlinear Partial Differential Equations By The Decomposition Method

Abstract

In this paper, the Adomian decomposition method (ADM) and the modified decomposition method (MDM) have been implemented for finding the solutions of Korteweg –de Vries and modified Korteweg –de Vries equations. The precision and speed of convergence of the numerical solution are verified through tables and illustrations via Mathematica software.

***Keywords**:* Adomian decomposition method, Modified Adomian decomposition Method, Generalized KdV equation, Cole-Hopf transformation solutions.

1. Introduction

The study of nonlinear partial differential equations (PDE's) has a crucial important role in various fields of mathematics and physics. The behavior of the nonlinearity hides most of interesting features of physical systems and can be studied with appropriate methods designed to tackle nonlinear phenomena. In last recent decades, there has been great development of literature in solving linear and nonlinear problems such as Bäcklund's transformation, Darboux's transformation, Cole-Hopf transformation, inverse scattering method [5,8,13,18,19]. Methods involving hyperbolic functions are used for finding the solitary travelling wave solutions for example Tanh method [24], extended tanh method [26], modified extended tanh method with Riccati equation, sech method, extended sech method [10,11], a mixed sech-tanh methods and so on.

The methods of decomposing has been the subject of extensive analytical and numerical studies. In particular, the Adomian decomposition method ADM [6,7] has emerged as a powerful technique for large and general class of linear and nonlinear ordinary differential equations (ODE's) as well as partial differential equations (PDE's), algebraic, integro-differential, differential-delay equations [6,7]. The method is efficient in solving initial-value or boundary value problems without unphysical restrictive assumptions such as linearization, perturbation and so forth. The ADM provides the solution in an infinite series that converges rapidly with elegant computable components [6,7]. In recent years a large amount of research work concerning the developing of the ADM is investigated. Wazwaz [21] proposed a powerful modification, namely the modified Decomposition Method (MDM), based on dividing the function f that arises from integrating the non-homogeneous term and using the provided conditions into two practical parts f_0, f_1. This modification may give the exact solution for nonlinear equations by using two iterations only and without the use of the so-called Adomian polynomials. However the criteria of being f consists of one term remains unsolved so far, and the success of this modification depends on the proper choice of the split. Following that Wazwaz and El-Sayed [23] presented a new modification that replaces the process of dividing the function by a series of infinite components. Recently, the Improved ADM (IADM) was introduced by Abassy [2,3] to overcome non-homogeneous problems with nonlinear invertible operators. ADM- Padé technique [1] is used to extend the domain of the solution and gain better accuracy.

Recently, Kaya and Aassila [16] applied the decomposition method to rational and travelling wave solutions of homogeneous and non-homogenous initial-value Korteweg –de Vries (KdV) equations in case of $n=1$. Abassy *et al.* [1] used the ADM- Padé technique to enhance the accuracy of rational and travelling wave solutions of homogeneous and initial-value KdV equations.

In this presented work, we shall consider the initial-value Korteweg –de Vries (KdV) equation [10.] in the form:

$$u_t + \alpha u u_x + \mu u_{xxx} = 0$$

$$u(x,0) = \frac{4\lambda^2 m^2}{\left(e^{\lambda c x} + m^2 e^{-\lambda c x}\right)^2},\ \lambda = \sqrt{\frac{\omega}{\mu c^3}},\ \alpha = 12\mu c^2$$

Where c is the wave number, ω is the wave speed and m is a modulus that $0 < m^2 \le 1$.

The initial-value modified Korteweg –de Vries (mKdV) equation [11] in the form:

$$u_t + \alpha u^2 u_x + \mu u_{xxx} = 0$$

$$u(x,0) = \sqrt{E}\left(e^{\frac{cx}{2}} + E e^{-\frac{cx}{2}}\right)^{-1}$$

Where E is a positive constant, c is the wave number.

2. Analysis of the used method

According to Adomian [6,7] KdV and mKdV equations in operator form:

$$L_t u + \alpha N u + \mu L_x u = 0$$

sides of (3), gives:

$$u(x,t) = u(x,0) - \alpha L_t^{-1} N u - \mu L_t^{-1} L_x u$$

The method assumes that the unknown function $u(x,t)$ is decomposed into an infinite series of the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

and the nonlinear operator $Nu = u^n u_x,\ n = 1,2$ into the infinite series of the so-called Adomian polynomials given by:

$$Nu = u^n u_x = \sum_{k=0}^{\infty} A_k$$

These polynomials can be defined as:

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0} \qquad ,k \geq 0$$

Evaluating the formula in
Error! Reference source not found., we list the first few

Adomian polynomials:

$$A_0 = u_0{}^n (u_0)_x$$

$$A_1 = u_0{}^{-1+n}(nu_1(u_0)_x + u_0(u_1)_x)$$

$$A_2 = \frac{1}{2}u_0{}^{-2+n}((-1+n)nu_1{}^2(u_0)_x + 2nu_0u_1(u_1)_x$$
$$+2u_0(nu_2(u_0)_x + u_0(u_2)_x))$$

$$A_3 = \frac{1}{6}u_0{}^{-3+n}(n(2-3n+n^2)u_1{}^3(u_0)_x + 3(-1+n)nu_0u_1{}^2(u_1)_x$$
$$+6nu_0u_1((-1+n)u_2(u_0)_x + u_0(u_2)_x) + 6u_0{}^2(nu_3(u_0)_x$$
$$+nu_2(u_1)_x + u_0(u_3)_x))$$

$$A_4 = \frac{1}{24}(((-3+n)(-2+n)(-1+n)nu_0{}^{-4+n}u_1{}^4$$
$$+12(-2+n)(-1+n)nu_0{}^{-3+n}u_1{}^2u_2$$
$$+12(-1+n)nu_0{}^{-2+n}u_2{}^2 + 24(-1+n)nu_0[x,t]^{-2+n}u_1u_3$$
$$+24nu_0{}^{-1+n}u_4)(u_0)_x + 4((-2+n)(-1+n)nu_0{}^{-3+n}u_1{}^3$$
$$+6(-1+n)nu_0{}^{-2+n}u_1u_2 + 6nu_0{}^{-1+n}u_3)(u_1)_x$$
$$+12((-1+n)nu_0{}^{-2+n}u_1{}^2 + 2nu_0{}^{-1+n}u_2)(u_2)_x$$
$$+24nu_0{}^{-1+n}u_1(u_3)_x + 24u_0{}^n(u_4)_x)$$

$\vdots$

and so on.

Substituting (5) and (6) into (4) we obtain the subsequent components by the following recursive equations :

$$u_0 = u(x,0)$$

$$u_{k+1} = -\alpha L_t^{-1}A_k - \mu L_t^{-1}L_x u_k \quad k \geq 0$$

Following the methodology of the modified decomposition method (MDM) [21], the zeroth components

$u_0 = f$ is possibly to be divided into two parts f_0, f_1 . Hence, the modified recursive scheme takes the form:

$$u_0 = f_0$$

$$u_1 = f_1 - \alpha L_t^{-1} A_0 - \mu L_t^{-1} L_x u_0$$

$$u_{k+1} = -\alpha L_t^{-1} A_k - \mu L_t^{-1} L_x u_k \quad k \ge 1$$

The practical solution of $u(x,t)$ will be the k^{th} –term approximation $\tilde{\imath}$ as:

$$\tilde{\imath} = \sum_{i=0}^{k-1} u_i(x,t) \quad ,k \ge 1$$

where $\tilde{\imath}$ approaches $u(x,t)$ as $k \to \infty$.

3. Numerical Results, Diagrams and Discussion

3.1. The Korteweg –de Vries (Kdv) equation

Considering the initial value problem in (1), a series expansion of few components of $u(x,t)$ using the ADM (8) and the modified ADM (9) is given by (respectively):

$$u(x,t) = u_0 + u_1 + \ldots$$

$$= \frac{4e^{2cx\lambda} m^2 \lambda^2}{(e^{2cx\lambda} + m^2)^2} + 32c e^{2cx\lambda} m^2 (e^{2cx\lambda} - m^2) t \lambda^5 (c^2 e^{4cx\lambda} \mu + c^2 m^4 \mu$$

$$+ e^{2cx\lambda} m^2 (\alpha - 10c^2 \mu))(e^{2cx\lambda} + m^2)^{-5} + \ldots$$

$$u(x,t) = u_0 + u_1 + \ldots$$

$$= \frac{4m^2 \lambda^2}{e^{2cx\lambda} + m^2} + -(e^{2cx\lambda} + m^2)^{-4} 4m^2 \lambda^2 (m^6 - 8c^3 e^{6cx\lambda} t \lambda^3 \mu$$

$$- 2e^{2cx\lambda} m^4 (-1 + 4ct\alpha\lambda^3 + 4c^3 t \lambda^3 \mu)$$

$$+ e^{4cx\lambda} m^2 (1 - 8ct\alpha\lambda^3 + 32c^3 t \lambda^3 \mu)) + \ldots$$

The analytical solution [10] takes the form:

$$u(x,t) = \frac{4\lambda^2 m^2}{\left(e^{\lambda cx-\omega t} + m^2 e^{-\lambda cx-\omega t}\right)^2}, \lambda = \sqrt{\frac{\omega}{\mu c^3}}, \alpha = 12\mu c^2$$

In order to numerically verify the accuracy of the used methods, we evaluate the values of the parameters to be assumed as $m = 0.5, c = 0.25, \ \mu = 1 \ \ \alpha = 0.75 \ \ , \omega = 0.015625$. A numerical comparison is presented in Table .1 along with a graphical representation of the k^{th} –term approximated solution with the aid of Mathematica. Values of percentage relative errors $\rho(\%)$ are compiled in Table .1, the errors calculated show that a high accurate approximated solution is obtained by using only three terms of the methods considered. Furthermore, Enhancing the precision by making errors grow smaller would require adding more terms to the decomposition series.

Table 1 The numerical results of $\tilde{\imath}$ approximations in (11) and (12) in comparison with the analytical solution (13) **.**

	$t = 2\times10$		$t = 2\times10^{-}$		$t = 2\times10^{-7}$	
x_i	ADM	MDM	ADM	MDM	ADM	MDM
-2.5	0.00127434	0.00111937	0.0000127577	0.0000127422	1.27579×10^{-7}	1.27577×10^{-7}
-1.5	0.00577066	0.00568399	0.0000577181	0.0000577095	5.77182×10^{-7}	5.77182×10^{-7}
-0.5	0.00963684	0.00959652	0.0000963744	0.0000963704	9.63745×10^{-7}	9.63744×10^{-7}
0.5	0.0126386	0.0126231	0.000126386	0.000126385	1.26386×10^{-6}	1.26386×10^{-6}
1.5	0.0147898	0.0147848	0.000147893	0.000147892	1.47893×10^{-6}	1.47893×10^{-6}
2.5	0.0162445	0.0162431	0.000162436	0.000162435	1.62435×10^{-6}	1.62435×10^{-6}

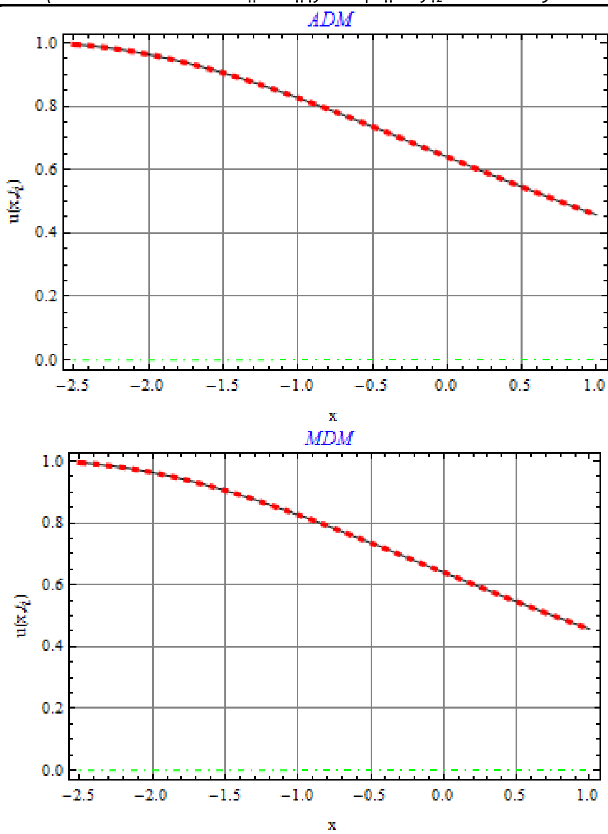
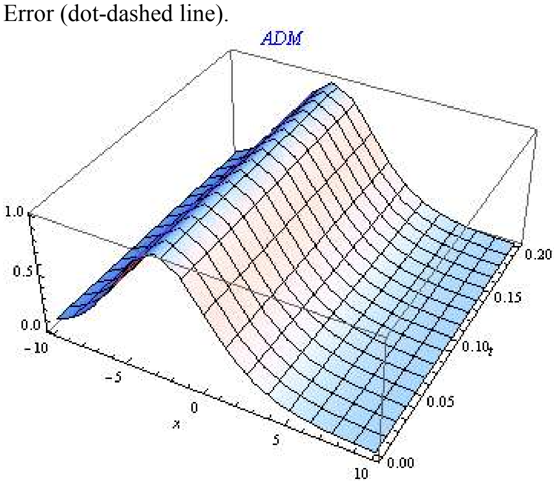


Figure 1 The corresponding decomposition method solution for (13) when $t = 2\times 10^{-3}$ sec. Exact $u(x,t)$ (solid line), Approximation $\tilde{u}$ (dashed Line), Error (dot-dashed line).



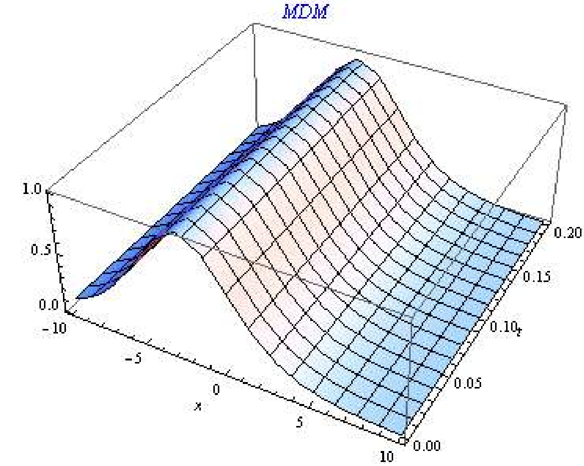


Figure 2 The behavior of The corresponding decomposition method surfaces for (13) when $t \in [0, 2 \times 10^{-1}]$ sec.

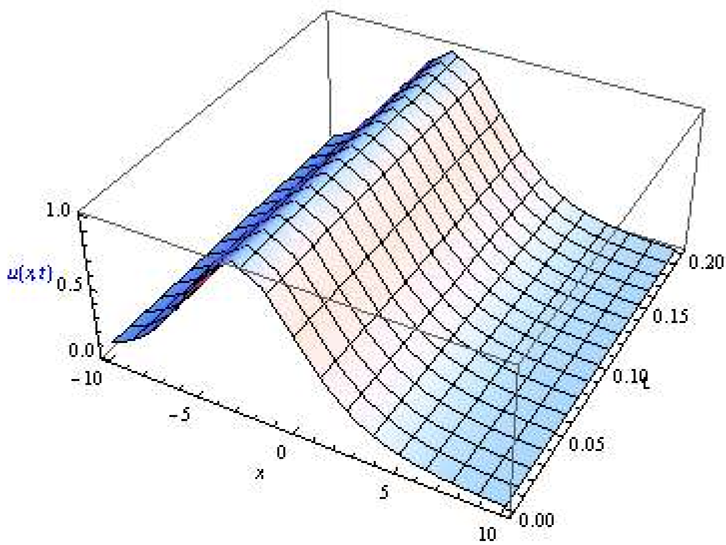


Figure 3 The Cole –Hopf surface (13).

3.2. The modified Korteweg –de Vries (mKdV) equation

Considering the initial value problem in (2), a series expansion of few components of $u(x,t)$ using the ADM (8) is given by :

$$u(x,t) = u_0 + u_1 + \ldots$$

$$= \frac{e^{\frac{cx}{2}}\sqrt{E}}{e^{cx}+E} + c e^{\frac{cx}{2}}(e^{cx} - E)\sqrt{E}t(c^2 e^{2cx}\mu + c^2E^2\mu$$

$$+ 2e^{cx}E(2\alpha - 11c^2\mu))8^{-1}(e^{cx} + E)^{-4} + \ldots$$

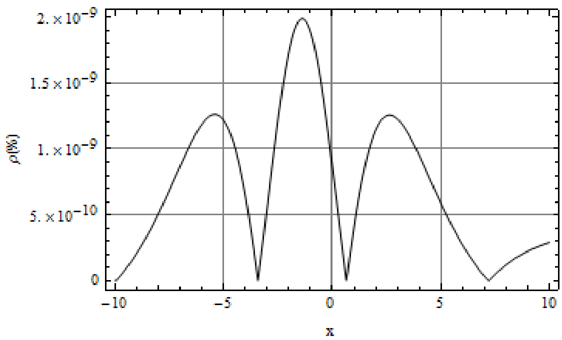
The analytical solution [11] is:

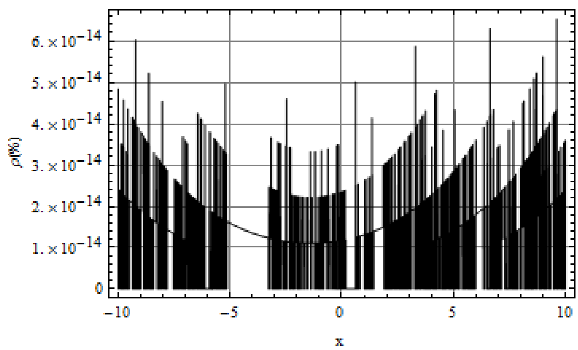
$$u(x,t) = \sqrt{E}\left(e^{\frac{1}{2}\left(cx-\frac{1}{4}\mu c^3 t\right)} + Ee^{-\frac{1}{2}\left(cx-\frac{1}{4}\mu c^3 t\right)} \right)^{-1}$$

with $E=0.5, c=0.5\ \mu=1$ a numerical comparison is presented in Table .2 along with a graphical representation of the *3rd* –term and *7th* –term approximated ADM' solution (for more details see [6,21]). With the aid of Mathematica the percentage relative errors for given values of time and space values are tabulated in Table 2. These errors proves the efficiency of the method considered, and for few terms of the decomposition series the solution converge rabidly. Finally, its concluded that the numerical results justify the advantage of the methodology of ADM as well as MDM.

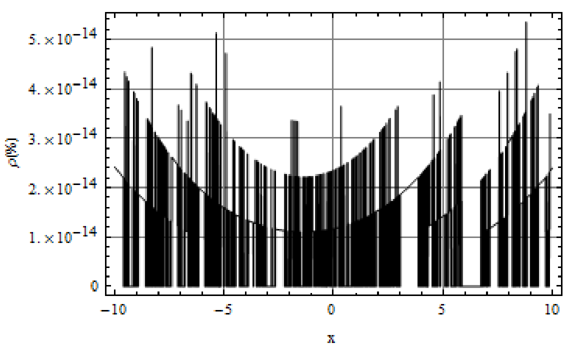
Table 2 The numerical results of $\tilde{i}$ (14) in comparison with the analytical solution.

	$t = 2\times10^{-1}$		$t = 2\times10^{-3}$		$t = 2\times10^{-5}$	
x_i	$k=3$	$k=7$	$k=3$	$k=7$	$k=3$	$k=7$
-10	7.02431×10^{-12}	1.21528×10^{-14}	0.	0.	1.21159×10^{-14}	1.21159×10^{-14}
-5	1.21713×10^{-9}	1.59855×10^{-14}	1.595×10^{-14}	1.595×10^{-14}	0.	0.
0	8.70945×10^{-10}	1.17635×10^{-14}	1.17756×10^{-14}	1.17756×10^{-14}	2.35514×10^{-14}	2.35514×10^{-14}
5	5.87186×10^{-10}	1.42217×10^{-14}	1.42623×10^{-14}	1.42623×10^{-14}	1.42627×10^{-14}	1.42627×10^{-14}
10	2.9182×10^{-10}	1.19579×10^{-14}	1.19947×10^{-14}	1.19947×10^{-14}	0.	0.





(a) **(b)**



(c)

Figure 4 The values of percentage relative error of the decomposition method solution $\tilde{\imath}$ when $t = 2\times10^{-1}, 2\times10^{-3}, 2\times10^{-5}$ sec (respectively **).**

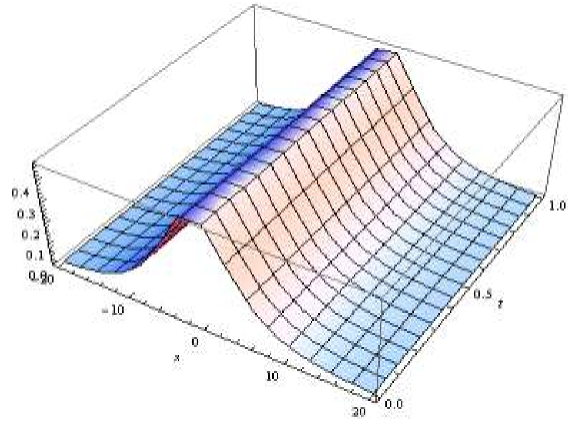


Figure 5 The behavior of The corresponding decomposition method surface for (15) when $t \in [0, 2\times 10^{-1}]$ **sec.**

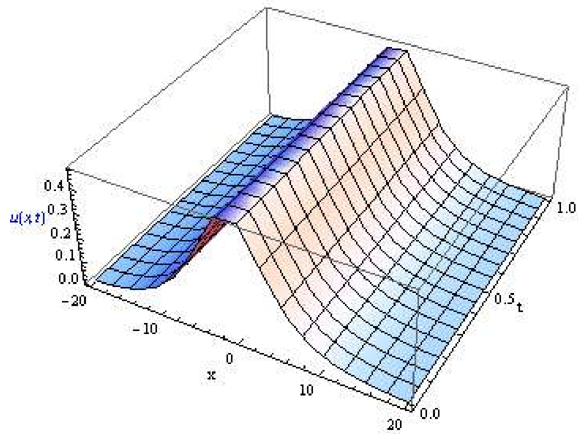


Figure 6 The Cole -Hopf Surface (15).

4. Conclusion

In this study, The ADM and the MDM have been successfully employed for finding the solution of Korteweg – de Vries and modified Korteweg –de Vries equations when $n = 1, 2$ with initial conditions. The results obtained are satisfactory for both ADM and MDM. Also, a high degree of accuracy is gained within few iterations and without restrictive assumptions. The numerical results show that the MDM is slightly more effective for the given IVP.

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