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WEAKLY CONVERGENCE OF THE 2D NAVIET-STOKES STRATIFIED SYSTEM

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ABSTRACT

In this paper, we study the inviscid limit for the Navier-Stokes stratified system in space dimension two with a real valued function F depend only on the temperature θ. Precisely we prove the weakly convergence in the space $L^{\infty}_{loc}(\mathbb{R}_+, L^2 \times L^2)$of the viscous solutions $(v_\nu, \theta_\nu)_{\nu>0}$ to the solution (v, θ) of the Euler stratified system as $\nu \to 0$.

Keywords: Navier-Stokes stratified system, Inviscid limit, Besov spaces, Dyadic decomposition.

الملخص

في هذا البحث ندرس تقارب منظومة معادلات نافيرستوكس الطبقية ثنائية البعد عندما لزوجة المائع ν تؤول إلى الصفر في الفضاء$L^{\infty}_{loc}(\mathbb{R}_+, L^2 \times L^2)$حيث أن الطرف الأيمن من معادلة نافيرستوكس هو دالة قيم حقيقية $F(\theta)$.

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1. INTRODUCTION

The Navier-Stokes stratified system for the incompressible fluid flows in $\mathbb{R}^2$ is of the form

$$\begin{cases} \partial_t v_\nu + v_\nu.\nabla v_\nu - \nu\Delta v_\nu + \nabla p_\nu = F(\theta_\nu)e_2, \\ \partial_t\theta_\nu + v_\nu.\nabla\theta_\nu - \Delta\theta_\nu = 0, \\ div\, v_\nu = 0, \quad (v_\nu,\theta_\nu)|_{t=0} = (v_0,\theta_0). \end{cases} \qquad (1.1)$$

Here the vector field $v_\nu = (v_\nu^1, v_\nu^2), v_\nu^j = v_\nu^j(x,t), j = 1,2, (x,t) \in \mathbb{R}^2 \times [0,\infty)$ Stands for the velocity of the field, and it is assumed to be divergence-free, The differential operator $v_\nu.\nabla$ is defined by :

$$v_\nu.\nabla = \sum_{j=1}^{d} v_\nu^j.\partial_j.$$

The scalar function $\theta_\nu(x,\mathrm{t})$ denotes the temperature and $p_\nu = p_\nu(x,t)$ is the scalar pressure. The parameter ν is the kinematic viscosity, the vector e_2 is given by $e_2 = (0,1)$ and the function F is a real valued function such that $F(0) = 0$. The operator $div\, v_\nu$is defined as

$$div\, v_\nu = \sum_{j=1}^{d} \partial_j v_\nu^j.$$

Note that the system (1.1) coincides with the classical incompressible Navier-Stokes equations when the initial temperature θ_0 is identically zero. It reads as follows :

$$\begin{cases} \partial_t v_\nu + v_\nu.\nabla v_\nu - \nu\Delta v_\nu + \nabla p_\nu = 0, \\ div\, v_\nu = 0 \\ v_\nu|_{t=0} = v_0 \end{cases} \qquad (1.2)$$

The mathematical theory of the incompressible Navier-Stokes equations (1.2) was initiated by Leray in [9]. He proved the global existence of a weak global solution of the system (1.2) in the energy space by using a compactness method. Nevertheless, the uniqueness of these solutions is only known for two spatial dimensions. A few decades later, in [7],Fujita and Kato proved local well-posedness in the critical Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, by using a fixed point argument and taking advantage of the time decay of the heat semi flow. The global existence of these solutions is only proved for small initial data and the question for large data remains an outstanding open problem . In [3], Bernicot and Keraani extended Yudovich's result to some class of initial vorticity in a Banach space which is strictly implicated betweenL^∞ and the space of bounded mean oscillations functions. Other results about this system are obtained.

Note that now if $F(\theta_\nu) = \theta_\nu\, e_2$, then the system (1.1) reduces to the system

$$\begin{cases} \partial_t v_\nu + v_\nu.\nabla v_\nu - \nu\Delta v_\nu + \nabla p_\nu = \theta_\nu\, e_2, \\ \partial_t\theta_\nu + v_\nu.\nabla\theta_\nu - \Delta\theta_\nu = 0, \\ div\, v_\nu = 0, \quad (v_\nu,\theta_\nu)|_{t=0} = (v_0,\theta_0), \end{cases} \qquad (1.3)$$

The global well-posedness for this system was studied by numerous authors and in a different functional spaces and the technic utilized to prove this result is to use the criterion of Beal-Kato and Majda in [4]. We mention that the paper of [13], where the author was studied the same result but for the system (1.1) in the case where $\nu = 0$ and $F(\theta) = (0,\theta)$.

Concerning the inviscid limit problem, that is the convergence of the viscous solutions $(v_\nu, \theta_\nu)_{\nu>0}$ to the solution of the incompressible Euler equation, we refer the reader to [6,8,10,12].
Now, we turn to the system (1.1), when the viscosity νequal to zero, we obtain the so-called Euler stratified system, this system is given by,

$$\begin{cases} \partial_t v + v.\nabla v + \nabla p = F(\theta) e_2, \\ \partial_t \theta + v.\nabla\theta - \Delta\theta = 0, \\ div\, v = 0, \quad (v, \theta)|_{t=0} = (v_0, \theta_0), \end{cases} \qquad (1.4)$$

The global well-posedness result for the system (1.4) was proved by Samira A. Sulaiman in[11]. In this paper, we prove that the viscous solution $(v_\nu, \theta_\nu)_{\nu>0}$ of the incompressible Navier-Stokes stratified system (1.1) is weakly converges to the solution (v, θ) of the incompressible Euler system (1.4). The main result of this paper is the following.

Theorem 1.1. Let $v_0 \in L^2(\mathbb{R}^2)$ be a divergence-free vector field ofvorticity ω, with$\omega_0 \in L^2$ and let $\theta_0 \in L^2 \cap L^\infty$ a real valued function. Let also $F \in C^1(\mathbb{R}, \mathbb{R}^2)$ such that $F(0) = 0$. Suppose also that (v_ν, θ_ν) respectively $(v, \theta)$are the solutions of the system (1.1) and the system (1.4) respectively. Then there exists a constant C depend only on $\|v_0\|_{L^2}$ and $\|\theta_0\|_{L^2 \cap L^\infty}$ such that for all $t \in [0, \infty)$, we have when $\nu \to 0$, that

$$\|v_\nu(t) - v(t)\|_{L^2(\mathbb{R}^2)} + \|\theta_\nu(t) - \theta(t)\|_{L^2(\mathbb{R}^2)} \to 0.$$

Remark 1.1. From the proof, we can conclude that

$$\|\mathrm{F}(\theta_\nu) - \mathrm{F}(\theta)\|_{L_t^\infty L^2} \leq \sqrt{\nu}\, C_0 log^2(1+t)\ f(t),$$

where$f(t)$ is a function depend only on t. This gives also that

$$\|\mathrm{F}(\theta_\nu) - \mathrm{F}(\theta)\|_{L_t^\infty L^2} \to 0 \text{as} \nu \to 0.$$

2. SOME USEFUL LEMMA

In this preliminary section, we introduce some basic notations and recall the definition of Besov space. We gives also some important results that will be used later. We will use the following notations :

- We denote by C any positive constant that will change from line to line and C_0 a real positive constant depending on the size of the initial data.
- For any positive constants Aand B, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.

Definition 2.1 (Schwartz space) We say that a function f in the Schwartz space Sif $f \in C^\infty(\mathbb{R}^d)$ and for all α and N there exist a constant $C_{N,\alpha}$ such that

$$|\partial^\alpha f(x)| \leq \frac{C_{N,\alpha}}{(1+|x|)^{-N}}.$$

To define Besov space, we need to define the dyadic decomposition of the full space $\mathbb{R}^2$ and to recall the Littlewood-Paley operators, see for example [2,5]. There exist two nonnegative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2/\{0\})$ such that :

$$\chi(\xi)+\sum_{q\geq 0}\varphi(2^{-q}\xi)=1,\qquad \forall\xi\in\mathbb{R}^2,$$

$$\sum_{q\in\mathbb{Z}}\varphi(\,2^{-q}\xi)=1, \forall\xi\in\mathbb{R}^2/\{0\},$$

$$|p-q|\geq 2\Rightarrow supp\,\varphi(2^{-p}.)\cap\ supp\,\varphi(2^{-q}.)=\phi,$$

$$q\geq 1\Rightarrow supp\,\chi\cap\ supp\,\varphi(2^{-q}.)=\phi.$$

Let $h=\mathcal{F}^{-1}\varphi$and $\bar{h}=\mathcal{F}^{-1}\chi$, the inhomogeneous frequency localization operators Δ_q and S_q are defined by :

$$\Delta_q f=\varphi(2^{-q}D)f,\qquad \forall q\geq 0.\qquad S_q f=\chi(2^{-q}D)f$$

$$\Delta_{-1}f=S_0 f,\qquad \Delta_q f=0\quad for\ q\leq -2.$$

The homogeneous operators are defined as follows

$$\forall q\in\mathbb{Z},\quad \dot{\Delta}_q f=\varphi(2^{-q}D)f,$$

We recall now the definition of Besov spaces, see[3,5].

Definition 2.2 (Besov space) Let $s\in\mathbb{R}$ and $1\leq p,r\leq\infty.$ The inhomogeneous Besov space $B^s_{p,r}$ defined by :

$$B^s_{p,r}=\left\{f\in\acute{S}(\mathbb{R}^2):\|f\|_{B^s_{p,r}}<\infty\right\},$$

Where S is the Schwartz space and

$$\|f\|_{B^s_{p,r}}:=\left(2^{qs}\left\|\Delta_q f\right\|_{L^p}\right)_{l^r}.$$

The homogeneous norm :

$$\|f\|_{B^s_{p,r}}:=\left(2^{qs}\left\|\dot{\Delta}_q f\right\|_{L^p}\right)_{l^r}.$$

We will need to the following proposition which its proof can be found in [3].

Proposition 2.1. Let (ω,θ) be a smooth solution of the following system

$$\begin{cases}\partial_t\omega+v.\nabla\omega+\nabla p=\partial_1\theta,\\ \partial_t\theta+v.\nabla\theta-\Delta\theta=0,\\ div\ v=0,\quad (\omega,\theta)|_{t=0}=(\omega_0,\theta_0).\end{cases}$$

Let also $\theta_0\in \mathrm{L}^1\cap\mathrm{L}^\mathrm{p}$ and $\omega_0\in\mathrm{L}^2\cap\mathrm{L}^\mathrm{p}$ with $2\leq \mathrm{p}\leq\infty$. Then for $\mathrm{t}\in\mathbb{R}_+$, we have

$$\|\omega(\mathrm{t})\|_{\mathrm{L^p}}+\|\nabla\theta\|_{\mathrm{L^1_tL^p}}\leq \mathrm{C_0}\log^{2-\frac{2}{\mathrm{p}}}(1+\mathrm{t}).$$

The proof of the following theorem can be found in [1,2].

Theorem 2.1 Let $F\in C^{[s]+2}$ with $F(0)=0$and $s\in[0,\infty]$. Assume that $\theta\in B^s_{p,r}\cap L^\infty$with $(p,r)\in[1,\infty]^2$, then $F\circ\theta\in B^s_{p,r}$ and satisfying :

$$\|F\circ\theta\|_{B^s_{p,r}}\leq C_s sup_{|x|<C\|\theta\|_{L^\infty}}\left\|F^{[s]+2}(x)\right\|_{L^\infty}\|\theta\|_{B^s_{p,r}}.$$

The following result is called as Gronwall's lemma. This lemma is very useful and well-known in the analysis, see [2] for a proof.

Lemma 2.1 (Gronwall's lemma)
Let f is a nonnegative continuous function on $[0,t]$, b is a real number and let B be a continuous function on $[0,t]$. Suppose also that :

$$f(t) \leq b + \int_0^t B(\tau) f(\tau) d\tau.$$

Then we have :

$$f(t) \leq b \, exp\left(\int_0^t B(\tau) d\tau \right).$$

3. PROOF OF THEOREM 1.1

To prove our main result Theorem 1.1, we first prove the following.

Proposition 3.1 Let$(\omega_\nu, \theta_\nu), \nu \geq 0$ be a smooth solution of the system

$$\begin{cases} \partial_t \omega_\nu + v_\nu . \nabla \omega_\nu - \nu \Delta \omega_\nu = \partial_1 (F_2(\theta_\nu)) - \partial_2 (F_1(\theta_\nu)), \\ \partial_t \theta_\nu + v_\nu . \nabla \theta_\nu - \Delta \theta_\nu = 0, \\ div \, v_\nu = 0, \quad (\omega_\nu, \theta_\nu)|_{t=0} = (\omega_0, \theta_0). \end{cases} \tag{3.1}$$

Suppose also that $\omega_0 \in L^2$, and $\theta_0 \in L^2 \cap L^\infty$.Let$F$ be a real valued function such that $F \in C^1(\mathbb{R}, \mathbb{R}^2)$, and $F(0) = 0$. Then we have

$$\|\omega_\nu(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2}.$$

Proof : From the first equation of (3.1), we have for every$\nu \geq 0$,

$$\partial_t \omega_\nu + v_\nu . \nabla \omega_\nu - \nu \Delta \omega_\nu = \partial_1 (F_2(\theta_\nu)) - \partial_2 (F_1(\theta_\nu))$$

$$= \grave{F}_2(\theta_\nu) \partial_1 \theta_\nu - \grave{F}_1(\theta_\nu) \partial_2 \theta_\nu.$$

Taking the L^2 norm and applying Holder inequality, we get :

$$\|\omega_\nu(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \sum_{i=1}^{2} \int_{\mathbb{R}^2} \left\| \grave{F}_i \circ \theta_\nu(\tau) \right\|_{L^\infty} \|\nabla \theta_\nu(\tau)\|_{L^2} d\tau \tag{3.2}$$

Now, since :

$$\left\| \grave{F}_i \circ \theta(t) \right\|_{L^\infty} = sup_{x,t} \left| \grave{F}_i(\theta(x,t)) \right|,$$

then using the embedding $B^0_{\infty,\infty} \hookrightarrow \dot{B}^0_{\infty,\infty} \hookrightarrow L^\infty$ and Theorem 2.1 , we get :

$$\left\| \grave{F}_i \circ \theta(t) \right\|_{L^\infty} \leq sup_{|x| \leq C \|\theta\|_{L^\infty}} \left| \grave{F}_i(x) \right|$$

$$\leq sup_{|x| \leq \|\theta_0\|_{L^\infty}} \left| \grave{F}_i(x) \right|.$$

As $F \in C^1(\mathbb{R}, \mathbb{R}^2)$, $\theta_0 \in L^2 \cap L^\infty$and applying Theorem 2.1,yields :

$$\left\|\tilde{F}_i \circ \theta_\nu(t)\right\|_{L^\infty} \leq sup_{|x|\leq \|\theta_0\|_{L^\infty}} \|\nabla F_i\|_{L^\infty} \leq C.$$

This gives in (3.2),that :

$$\|\omega_\nu(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + C\|\nabla\theta_\nu\|_{L^1_t L^2} \qquad (3.3)$$

It remains then to estimate $\|\nabla\theta_\nu\|_{L^1_t L^2}$. For this purpose, we take the scalar product of the second equation of (3.1) with θin L^2space. Then the incompressibility condition $div\ v_\nu = 0$ leads to the following energy estimate :

$$\frac{1}{2}\frac{d}{dt}\|\theta_\nu(t)\|_{L^2}^2 + \|\nabla\theta_\nu(t)\|_{L^2}^2 = 0.$$

Then :

$$\frac{d}{dt}\|\theta_\nu(t)\|_{L^2}^2 + 2\|\nabla\theta_\nu(t)\|_{L^2}^2 = 0.$$

Integrating in time, we get :

$$\|\theta_\nu(t)\|_{L^2}^2 + 2\|\nabla\theta_\nu\|_{L^1_t L^2}^2 = \|\theta_0\|_{L^2}^2.$$

Therefore,

$$\|\nabla\theta_\nu\|_{L^1_t L^2}^2 \leq \|\theta_0\|_{L^2}^2.$$

This implies that :

$$\|\nabla\theta_\nu\|_{L^1_t L^2} \leq \|\theta_0\|_{L^2}.$$

Also

$$\|\theta_\nu(t)\|_{L^2} \leq \|\theta_0\|_{L^2} \qquad (3.4)$$

This gives in (3.3) for $\nu \geq 0$that :

$$\|\omega_\nu(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \|\theta_0\|_{L^2}.$$

Proposition 3.2

Let $v_0 \in L^2$, and $\theta_0 \in L^2 \cap L^\infty$.Let also$F$ be a real valued function such that $F \in C^1(\mathbb{R},\mathbb{R}^2)$, and $F(0) = 0$. Then we have

$$\|v_\nu(t)\|_{L^2} \leq C_0 e^{Ct},$$

whereC_0 depend only on L^2 norm of v_0 and $L^2 \cap L^\infty$ norm of θ_0.

Proof :

Taking the L^2inner product of the first equation of (1.1) with v_ν and using Holder inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 + \nu\|\nabla v_\nu(t)\|_{L^2}^2 \leq \|F(\theta_\nu)\|_{L^2}\|v_\nu(t)\|_{L^2}$$

This gives that,

$$\frac{1}{2}\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 \leq \|F(\theta_\nu)\|_{L^2}\|v_\nu(t)\|_{L^2}$$

Using now the inequality, $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, for any a and b, we find

$$\frac{1}{2}\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 \le C\|\mathrm{F}(\theta_\nu)\|_{L^2}^2 + C\|v_\nu(t)\|_{L^2}^2 \qquad (3.5)$$

To estimate $\|\mathrm{F}(\theta_\nu)\|_{L^2}$, we use Taylor formula with $F(0) = 0$, then

$$F(\theta_\nu) = \theta_\nu \int_0^1 \acute{F}(\tau\,\theta_\nu)d\tau$$

This gives that,

$$\|\mathrm{F}(\theta_\nu)\|_{L^2} \le \|\theta_\nu\|_{L^2} \int_0^1 \left\|\acute{F}(\tau\,\theta_\nu)\right\|_{L^\infty} d\tau$$

Now, since :

$$\left\|\acute{F}(\tau\,\theta_\nu)\right\|_{L^\infty} \le sup_{|x|\le C\|\theta\|_{L^\infty}} \left|\acute{F}(x)\right|,$$

$$\le sup_{|x|\le C\|\theta_0\|_{L^\infty}} \left|\acute{F}(x)\right|$$

As $F \in C^1(\mathbb{R}, \mathbb{R}^2), \theta_0 \in L^2 \cap L^\infty$and applying Theorem 2.1,yields :

$$\left\|\acute{F}(\tau\,\theta_\nu)\right\|_{L^\infty} \le sup_{|x|\le\|\theta_0\|_{L^\infty}} \left|\acute{F}(x)\right| \le C.$$

Therefore, we obtain

$$\|\mathrm{F}(\theta_\nu)\|_{L^2} \le C\|\theta_\nu\|_{L^2} \le C\|\theta_0\|_{L^2},$$

where we have used (3.4). Putting now the above inequality in (3.5), we get

$$\frac{1}{2}\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 \le C\|\theta_0\|_{L^2}^2 + C\|v_\nu(t)\|_{L^2}^2$$

Integrating in time the above inequality, we find

$$\|v_\nu(t)\|_{L^2}^2 \le C\|v_0\|_{L^2}^2 + C\|\theta_0\|_{L^2}^2 t + C\int_0^t \|v_\nu(\tau)\|_{L^2}^2 d\tau$$

Using Lemma 2.1 of Gronwall, we get

Therefore

$$\|v_\nu(t)\|_{L^2} \le C_0 e^{\,Ct}.$$

■

Proposition 3.3

Under the same hypothesis of the previous propositions, we have

$$\|v_\nu(t) - v(t)\|_{L^2} + \|\theta_\nu(t) - \theta\,(t)\|_{L^2} \le \sqrt{\nu}\,C_0\;g(t)\;e^{C_0\sqrt{t}},$$

with$g$ is a function depend explicitly only on t.

Proof : let $V_\nu = v_\nu - v$, $\Theta_\nu = \theta_\nu - \theta$, and $\Pi_\nu = p_\nu - p$. Then we have

$$\begin{cases} \partial_t V_\nu + v_\nu.\nabla V_\nu + V_\nu.\nabla v - \nu\Delta V_\nu + \nabla \Pi_\nu = \nu\Delta v + (F(\theta_\nu) - F(\theta))e_2, \\ \partial_t \Theta_\nu + v_\nu.\nabla\Theta_\nu - \Delta\Theta_\nu = -V_\nu \nabla\theta, \\ div\, V_\nu = 0, \quad (V_\nu, \Theta_\nu)|_{t=0} = 0, \end{cases} \tag{3.6}$$

Taking the L^2inner product of the first equation of (3.6) with V_ν and using Holder inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|_{L^2}^2 + \nu\|\nabla V_\nu(t)\|_{L^2}^2 \le \nu\,\|\nabla v\|_{L^2}\|\nabla V_\nu\|_{L^2} + \|\nabla v\|_{L^2}\|V_\nu(t)\|_{L^2}^2 + \|F(\theta_\nu) - F(\theta)\|_{L^2}\|V_\nu\|_{L^2}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|_{L^2}^2 \le \nu\|\nabla v\|_{L^2}\|\nabla V_\nu\|_{L^2} + \|\nabla v\|_{L^2}\|V_\nu(t)\|_{L^2}^2 + \|F(\theta_\nu) - F(\theta)\|_{L^2}\|V_\nu(t)\|_{L^2}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|_{L^2}^2 \le I + II + III \tag{3.7}$$

For the first term, we have

$$I = \nu\|\nabla v\|_{L^2}\|\nabla V_\nu\|_{L^2} \le \nu\|\omega\|_{L^2}(\|\omega_\nu\|_{L^2} + \|\omega\|_{L^2})$$

Using now Proposition 3.1, we get

$$I \le \nu(\|\omega_0\|_{L^2} + \|\theta_0\|_{L^2}) \tag{3.8}$$

For II, we immediate have

$$\begin{aligned} II &= \|\nabla v(t)\|_{L^2}\|V_\nu(t)\|_{L^2}^2 \\ &\le \|\omega(t)\|_{L^2}\|V_\nu(t)\|_{L^2}^2 \end{aligned}$$

Using now Proposition 3.1 for$\nu = 0$, we get

$$II \le (\|\omega_0\|_{L^2} + \|\theta_0\|_{L^2})\|V_\nu(t)\|_{L^2}^2 \tag{3.9}$$

For the last term, we need to estimate $\|F(\theta_\nu) - F(\theta)\|_{L^2}$, we use Taylor formula of order 1, we have

$$\mathrm{F}(\theta_\nu) - \mathrm{F}(\theta) = (\theta_\nu - \theta)\int_0^1 F'\big(\theta + \tau(\theta_\nu - \theta)\big)d\tau.$$

This gives

$$\|\mathrm{F}(\theta_\nu) - \mathrm{F}(\theta)\|_{L^2} \le \|\theta_\nu - \theta\|_{L^2}\int_0^1 \big\|F'\big(\theta + \tau(\theta_\nu - \theta)\big)\big\|_{L^\infty} d\tau.$$

We write now,

$$\left\|\acute{F}(\theta+\tau(\theta_\nu-\theta))\right\|_{L^\infty} \leq sup_{|x|\leq\|\theta_0\|_{L^\infty}}\left|\grave{F}_i(x)\right| \leq C.$$

Thus, we obtain

$$\|\mathrm{F}(\theta_\nu)-\mathrm{F}(\theta)\|_{L^2} \leq C\;\|\theta_\nu-\theta\|_{L^2}$$

It remains then to estimate $\|\theta_\nu-\theta\|_{L^2,}$ for this we have

$$\|\Theta_\nu\|_{L^2} = \|\theta_\nu-\theta\|_{L^2} \leq \int_0^t \|V_\nu.\nabla\theta(\tau)\|_{L^2}d\tau \leq \|V_\nu\|_{L_t^\infty L^2}\|\nabla\theta\|_{L_t^1 L^\infty}$$

$$\leq C_0 log^2(1+t)\|V_\nu\|_{L_t^\infty L^2},$$

where we have used Proposition 2.1. This gives that

$$\|\mathrm{F}(\theta_\nu)-\mathrm{F}(\theta)\|_{L^2} \leq C\;\|\theta_\nu-\theta\|_{L^2}$$

$$\leq C_0 log^2(1+t)\|V_\nu\|_{L_t^\infty L^2}.$$

Therefore the last term III is estimated by

$$III \leq C_0 log^2(1+t)\|V_\nu\|^2_{L_t^\infty L^2} \qquad (3.10)$$

Putting now (3.8), (3.9) and (3.10) into (3.7), we find

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|^2_{L^2} \leq \nu(\|\omega_0\|_{L^2}+\|\theta_0\|_{L^2}) + [C_0 log^2(1+t)+\|\omega_0\|_{L^2}+\|\theta_0\|_{L^2}]\|V_\nu\|^2_{L_t^\infty L^2}.$$

Therefore we can write the last inequality as :

$$\frac{1}{2}\frac{d}{dt}\|V_\nu(t)\|^2_{L^2} \leq \nu C_0 + C_0 log^2(1+t)\|V_\nu\|^2_{L_t^\infty L^2}$$

Integrating in time the above inequality, we get to

$$\|V_\nu(t)\|^2_{L_t^\infty L^2} \leq \nu\, C_0\, t + C_0\int_0^t log^2(1+\tau)\|V_\nu\|^2_{L_\tau^\infty L^2}d\tau.$$

Lemma of Gronwall 2.1, yields to

$$\|V_\nu(t)\|^2_{L_t^\infty L^2} \leq \nu\, C_0\, t\;\exp\,[\,C_0\int_0^t log^2(1+\tau)\,d\tau].$$

This gives that

$$\|V_\nu(t)\|^2_{L_t^\infty L^2} \leq \nu\, C_0 f_1(t),$$

where

$$f_1(t) := t\exp\left[C_0\int_0^t log^2(1+\tau)\,d\tau\right].$$

Therefore, we obtain

$$\|V_\nu(t)\|_{L_t^\infty L^2} \leq \sqrt{\nu}\, C_0\, f(t),$$

with$f(t) := \sqrt{f_1(t)}$. This gives that

$$\|\Theta_\nu\|_{L_t^\infty L^2} \leq C_0 log^2(1+t)\|V_\nu\|_{L_t^\infty L^2}$$

$$\leq \sqrt{\nu} C_0 log^2(1+t) f(t).$$

Therefore

$$\|\mathrm{F}(\theta_\nu) - \mathrm{F}(\theta)\|_{L_t^\infty L^2} \leq \sqrt{\nu} C_0 log^2(1+t)\, f(t).$$

Finally, we get

$$\|V_\nu(t)\|_{L_t^\infty L^2} + \|\Theta_\nu(t)\|_{L_t^\infty L^2} \leq \sqrt{\nu}\, C_0\, g(t),$$

with$g(t) \coloneqq\ log^2(1+t) f(t)$. Putting now $\nu \to 0$, we get

$$\|V_\nu(t)\|_{L^2} + \|\Theta_\nu(t)\|_{L^2} \to 0,$$

which is the desired result.

4.CONCLUSIONS

We have proved that the solution $(v_\nu, \theta_\nu)_{\nu>0}$of the Navier-Stokes stratified system is weakly converges to the solution (v, θ) of the Euler-stratified system in space dimension two. We used in our proof the Taylor formula at order 1 of the real valued function F.

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