التقريبات شبه التبولوجية لمجموعات الإستقراب

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الملخص العربي :

في هذا البحث تم إستحداث مؤثرين جديدين :
(Prelower and preupper approximations)
وذلك بإستخدام بنية توبولوجية مولدة بعلاقة عامة تعتمد على المفاهيم الخاصة بالمجموعات
قريبة الفتح (Perinterior) وقريبة الغلق (preclosure) ، وتم دراسة خصائصهم وتقديم العديد من الأمثلة. أيضاً في هذه البحث تم تمييزالفراغ شبة التقاربي(preapproximation space) بمناطق مختلفة بإستخدام هذه المفاهيم وتم دراسة خصائصهم والعلاقة بينهم وبين المفاهيم الخاصة بفراغ باولاك (Pawlak space) وتم استحداث أنواع جديدة من تساوي و إحتواء مجموعات الإستقراب تعتمد علي مؤثري الغلق والفتح المولدين من علاقة عامة والمؤثرين (Perlower and preupper approximation) وتم دراسة خصائصهم والعلاقة بينهم وكذلك تم في هذا البحث زيادة عدد حدود مجموعات الإستقراب الي خمس حدود بدلا من ثلاثة عند باولاك ودراسة خصائصهما وأثبتنا أنه يوجد تجزيء لمنطقة الحدود boundary region)

Pretopological Approximations of Rough Sets

Abstract

In this paper, we generalize the subsystem based definition of rough set using a topological structure generated from a general relation. We construct new approximations based on topological notions of preclosure and preinterior, Then their properties are studed. The number of possible membership relations are enlarged, The properties of these approximation are discussed. Also we introduce new types of rough definability and

undefinability, based on the notions of prelower and preupper

approximations.

Introduction

Topology scionce is a mathematical tool to study information

systems and rough sets [2,8]. Topologically, lower and upper approximations in the original rough set model are respectively the closure and interior with respect to aspecial type of topological structure in which every open set is closed [8].The subsystem based formulation provides an important interpretation of the rough set theory [4,10]. It allows us to study the rough set theory on the contexts of many algebraic systems. This leads naturally to the generalization of rough set approximations. In the last three decades of the twentieth century, many sorts of near closure and near interior operators have been introduced in the theory of topological spaces but to our knowledge many of them have not been applied in the context of rough sets [2,8]. Preclosure and preinterior are examples of near closure and near interior operators. The construction of preclosure and preinterior of a subset X depends on the deviation between the interior of the closure of X ($int\ cl\ X$) and the closure of the interior of X ($cl\ int\ X$).

The aim of this paper is to generalize the subsystem based definition of rough sets by using a topological structure generated from a general relation. We construct another approximations (prelower and preupper approximations) based on the well known topological notion of

preclosures and preinteriors, Then wearc study their properties.The other parts of this paper are arranged as follows. In Section (1&2) we present a new approximation by using a general topological structure, we introduce

new notions of approximations called a prelower and a preupper

approximation and study their properties. We show that the class of pre exact sets contains the class of exact sets which increases the accuracy. Also, we characterize the approximation space with different regions by representing certain concept of interest and study the properties also the relationships between them. Finally in Sections (3) and (4) the number of possible membership relations by using these new concepts are enlarged.

1- Preapproximations concept:

Definition 1.1 [9]

Let (U,T) be a topological space. The subset $X \subseteq U$ is called

preopen (briefly, $P-$ open) if $X \subseteq \overline{\overline{X}}^{\circ}$. The complement of a $P-$ open set is called P-closed, the family of all P-open set of (U,T) is denoted by $PO(U)$, the family of all $P-$ closed set of (U,T) is denoted by

$PC(U)$.

Example 1.1

Let $U=\{a,b,c\}$, $R=\{(a,a),(a,b),(b,b),(c,a),(c,c)\}$,

$S=\big\{\{a,b\},\{b\},\{a,c\}\big\}$, $\beta=\big\{U,\Phi,\{a\},\{b\},\{a,b\},\{a,c\}\big\}$,

$T=\big\{U,\Phi,\{a\},\{b\},\{a,b\},\{a,c\}\big\}$,

$T^c=\big\{U,\Phi,\{b\},\{b,c\},\{c\},\{a,c\}\big\}$. Then the classes of P-open sets is $PO(U)=\big\{U,\Phi,\{a\},\{b\},\{a,b\}\big\}$,

$P-$ closed set is $PC(U)=\big\{U,\Phi,\{c\},\{a,c\},\{b,c\}\big\}$.

Definition 1.2 $[\mathbf{1},\mathbf{5}]$

Let (U,T) be a topological space, and $X \subseteq U$. Then the $P-closure$ of X is denoted by $X^{\overline{P}}$, and is defined by

$X^{\overline{P}} = \cap \{F \subseteq U: X \subseteq F, F\ is\ a\ P-closed\ set\}$.

Definition 1.3 $[\mathbf{1},\mathbf{5}]$

Let (U,T) be a topological space, and $X \subseteq U$.Then the $P-\ interior$ of X is denoted by $X^{P^{\circ}}$, and is defined by

$X^{P^{\circ}} = \cup \{G \subseteq U: G \subseteq X, G\ is\ a\ P-open\ set\}$.

Proposition 1.1 [6]

Let (U,T) be a topological space, and $X \subseteq U$, then $X^{P^{\circ}} = X \cap \overline{X}^{\circ}$.

Proposition 1.2 [6]

Let (U,T) be a topological space, and $X \subseteq U$, then $X^{\overline{P}} = X \cup \overline{X^{\circ}}$.

Definition 1.4 [1]

Let (U,T) be a topological space, and $X \subseteq U$. Then the $P-boundary\ region$ of X is denoted by $BN(X^P)$ and is defined by $BN(X^P) = X^{\overline{P}} - X^{P^{\circ}}$.

Definition 1.5

Let (U,T) be a topological space, $X \subseteq U$. Then the $P-exterior$ of X $(EXT(X^P))$ is defined by $EXT(X^P) = U - X^{\overline{P}}$.

Now we present some new definitions and results as following:

Definition 1.6

Let $(U,\mathbb{R})$ be a general knowledge base, $X \subseteq U$, Then the $P-upper$ approximation of X ($\overline{P}(X)$) is defined by $\overline{P}(X) = X^{\overline{P}}$.

Definition 1.7

Let $(U,\mathbb{R})$ be a general knowledge base, $X \subseteq U$, Then $P-lower$ approximation of X ($\underline{P}(X)$) is defined by $\underline{P}(X) = X^{P^{\circ}}$.

Proposition 1.3

Let$(U,\mathbb{R})$ be a general knowledge base, $X \subseteq U$, then $\underline{R}(X) \subseteq \underline{P}(X) \subseteq X \subseteq \overline{P}(X) \subseteq \overline{R}(X)$.

Proof

$\underline{R}(X) = X^{\circ} = \cup\{G \in T: G \subseteq X\} \subseteq \cup\{G \in PO(X): G \subseteq X\}$, since

$T \subseteq PO(X) = X^{P^{\circ}} = \underline{P}(X) \subseteq X.$ $\longrightarrow$(1)

And $\overline{R}(X) = \overline{X} = \bigcap\{F \in T^C : X \subseteq F\} \supseteq \bigcap\{F \in PC(X) : X \subseteq F\}$, since

$T^C \subseteq PC(X) = \overline{X^P} = \overline{P}(X) \supseteq X. \quad \longrightarrow (2)$

From (1) & (2) we get, $\underline{R}(X) \subseteq \underline{P}(X) \subseteq X \subseteq \overline{P}(X) \subseteq \overline{R}(X)$.

The following example illustrates the existence of a space in which the class of $P-$ open sets form P-approximation space.

Example 1.5

Let $U = \{a, b, c, d\}$, and the relation

$R = \{(c, a), (c, b), (c, c), (d, b), (d, c)\}$, $S = \{\{a, b, c\}, \{b, c\}\}$,

$\beta = \{U, \Phi, \{a, b, c\}, \{b, c\}\}$, hence the topology generated by the relation is $T = \{U, \Phi, \{a, b, c\}, \{b, c\}\}$, $T^C = \{U, \Phi, \{d\}, \{a, d\}\}$, the class of

$P-$ open set is

$$PO(U) = \begin{Bmatrix} U, \Phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \\ \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \end{Bmatrix},$$

Thus $(U, PO(U))$ is preapproximation space.

The following example shows that the class of $P-$ open sets coincides with the topology if the topology is quasi discrete topology.

Example 1.6

Let $U = \{a, b, c, d\}$, $R = \{(c, b), (c, c), (d, a), (d, d)\}$,

$S = \{\{b, c\}, \{a, d\}\}$, $\beta = \{U, \Phi, \{b, c\}, \{a, d\}\}$, $T = \{U, \Phi, \{b, c\}, \{a, d\}\}$,

$T^C = \{U, \Phi, \{a, d\}, \{b, c\}\}$, the class of sets $P-$ open is

$$PO(U) \begin{Bmatrix} U, \Phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \\ \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \end{Bmatrix}.$$

Proposition 1.4

Let$(U,\mathbb{R})$ be a general knowledge base, and $X \subseteq U$, $R \in \mathbb{R}$, then the prelower and preupper approximation have the following properties:

1- $\underline{P}(\Phi) = \overline{P}(\Phi) = \Phi$, $\underline{P}(U) = \overline{P}(U) = U$.

2- If $X \subseteq Y \implies \underline{P}(X) \subseteq \underline{P}(Y)$ & $\overline{P}(X) \subseteq \overline{P}(Y)$.

3- $\underline{P}(X \cup Y) \supseteq \underline{P}(X) \cup \underline{P}(Y)$.

4- $\underline{P}(X \cap Y) \subseteq \underline{P}(X) \cap \underline{P}(Y)$.

5- $\overline{P}(X \cup Y) \supseteq \overline{P}(X) \cup \overline{P}(Y)$.

6- $\overline{P}(X \cap Y) \subseteq \underline{P}(X) \cap \underline{P}(Y)$.

7- $\underline{P}(X^C) = \left(\overline{P}(X)\right)^C, \overline{P}(X^C) = \left(\underline{P}(X)\right)^C$.

8- $\underline{P}\left(\underline{P}(X)\right) = \underline{P}(X)$, $\overline{P}\left(\overline{P}(X)\right) = \overline{P}(X)$.

Proof

1- $\underline{P}(\Phi) = \Phi \cup \underline{R}\left(\overline{R}(\Phi)\right) = \Phi \cup \underline{R}(\Phi) = \Phi, \overline{P}(\Phi)$

$= \Phi \cap \overline{R}\left(\underline{R}(\Phi)\right) = \Phi.$

And $\underline{P}(U) = U \cup \underline{R}\left(\overline{R}(U)\right) = U = \overline{P}(U)$.

2- If $X \subseteq Y \implies \overline{R}(X) \subseteq \overline{R}(Y) \implies \underline{R}\overline{R}(X) \subseteq \underline{R}\overline{R}(Y) \implies$

$X \cap \underline{R}\overline{R}(X) \subseteq Y \cap \underline{R}\overline{R}(Y)$. Then $\underline{P}(X) \subseteq \underline{P}(Y)$.

And $\overline{P}(X) \subseteq \overline{P}(Y)$ is obvious.

3- Since $X \subseteq X \cup Y$ & $Y \subseteq X \cup Y$, by (2) we have $\underline{P}(X) \subseteq \underline{P}(X \cup Y)$&

$\underline{P}(Y) \subseteq \underline{P}(X \cup Y)$, then $\underline{P}(X) \cup \underline{P}(Y) \subseteq \underline{P}(X \cup Y)$.

4- Since $X \cap Y \subseteq X$ & $X \cap Y \subseteq Y$, by (2) we have $\underline{P}(X \cap Y) \subseteq \underline{P}(X)$&

$\underline{P}(X \cap Y) \subseteq \underline{P}(Y)$. Hence $\underline{P}(X \cap Y) \subseteq \underline{P}(X) \cap \underline{P}(Y)$.

5- $\overline{P}(X \cup Y) = (X \cup Y) \cup \overline{R}\left(\underline{R}(X \cup Y)\right)$

$\supseteq (X \cup Y) \cup \overline{R}\left[\underline{R}(X) \cup \underline{R}(Y)\right]$

since $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$

$=(X \cup Y) \cup \left[\overline{R}\underline{R}(X) \cup \overline{R}\underline{R}(Y)\right]$

$= \left[(X \cup Y) \cup \overline{R}\underline{R}(X)\right] \cup \left[(X \cup Y) \cup \overline{R}\underline{R}(Y)\right]$

$\supseteq \left(X \cup \overline{R}\underline{R}(X)\right) \cup \left(Y \cup \overline{R}\underline{R}(Y)\right) = \overline{P}(X) \cup \overline{P}(Y)$.

6- If $X \cap Y \subseteq X \implies \underline{P}(X \cap Y) \subseteq \underline{P}(X) \longrightarrow (1)$ & if $X \cap Y \subseteq Y \implies$

$\underline{P}(X \cap Y) \subseteq \underline{P}(Y) \longrightarrow (2)$

From (1) & (2) we have $\underline{P}(X \cap Y) \subseteq \underline{P}(X) \cap \underline{P}(Y)$.

7- $\underline{P}(X^C) = X^C \cap \underline{R}\overline{R}(X^C) = X^C \cap \underline{R}\left(\underline{R}(X)\right)^C = X^C \cap \left(\overline{R}\underline{R}(X)\right)^C$

$= \left(X \cup \overline{R}\underline{R}(X)\right)^C = \left(\overline{P}(X)\right)^C$.

Also, $\overline{P}(X^C) = X^C \cup \overline{R}\underline{R}(X^C) = X^C \cup \overline{R}\left(\overline{R}(X)\right)^C = X^C \cup \left(\underline{R}\overline{R}(X)\right)^C$

$= \left(X \cap \underline{R}\overline{R}(X)\right)^C = \left(\underline{P}(X)\right)^C$.

8- $\underline{P}\left(\underline{P}(X)\right) = \underline{P}(X) \cap \left(\underline{R}\overline{R}\left(\underline{P}(X)\right)\right)$

$= \left(X \cap \underline{R}\overline{R}(X)\right) \cap \left(\underline{R}\left(\overline{R}\left(X \cap \underline{R}\overline{R}(X)\right)\right)\right)$.

Since $\overline{R}(X) \subseteq \overline{R}\left(X \cap \underline{R}\overline{R}(X)\right)$, then $\underline{R}\overline{R}(X) \subseteq \underline{R}\overline{R}\left(X \cap \underline{R}\overline{R}(X)\right)$, hence

$\implies \left(X \cap \underline{R}\overline{R}(X)\right) = \underline{P}(X) \subseteq \left(X \cap \underline{R}\overline{R}(X)\right) \cap \underline{R}\left(\overline{R}\left(X \cap \underline{R}\overline{R}(X)\right)\right)$

$\implies \underline{P}\left(\underline{P}(X)\right) = \underline{P}(X)$. Also, $\overline{P}\left(\overline{P}(X)\right) = \overline{P}\left(\underline{P}(X^C)\right)^C = \left(\underline{P}\left(\underline{P}(X^C)\right)\right)^C$

$= \underline{P}(X^C)^C = \overline{P}(X)$. Hence $\overline{P}\left(\overline{P}(X)\right) = \overline{P}(X)$.

Proposition 1.5

Let $(U, \mathbb{R})$ be a general knowledge base and $X \subseteq U$ then the following statements are true

1- $\underline{R}(X) \subseteq \underline{P}(X) \subseteq X \subseteq \overline{P}(X) \subseteq \overline{R}(X)$.

2- $\underline{R}\left(\underline{P}(X)\right) = \underline{P}\left(\underline{R}(X)\right) = \underline{R}(X)$.

3- $\overline{R}\left(\overline{P}(X)\right) = \overline{P}(\overline{R}(X)) = \overline{R}(X)$.

Proof

1- Since $\underline{R}(X) \subseteq X$ & $\underline{R}(X) \subseteq \underline{R}\overline{R}(X)$, then

$\underline{R}(X) \subseteq X \cap \underline{R}\overline{R}(X) = \underline{P}(X) \Longrightarrow \underline{R}(X) \subseteq \underline{P}(X)$. Also, since

$X \subseteq X \cup \overline{R}\underline{R}(X)$ & $\overline{R}\underline{R}(X) \subseteq \overline{R}(X)$, then $X \cup \overline{R}\underline{R}(X) = \overline{P}(X) \subseteq \overline{R}(X)$.

Or $\underline{R}(X) = X^{\circ} = \bigcup\{G \in T: G \subseteq X\} \subseteq \bigcup\{G \in PO(X): G \subseteq X\}$,

since $T \subseteq PO(X) = \underline{P}(X) \subseteq X \longrightarrow (1)$

$\overline{R}(X) = \overline{X} = \bigcap\{F \in T^C: X \subseteq F\} \supseteq \bigcap\{F \in PC(X): X \subseteq F\}$.

Since $T^C \subseteq PC(X) = \overline{P}(X) \supseteq X \longrightarrow (2)$. From (1) & (2) we have

$\underline{R}(X) \subseteq \underline{P}(X) \subseteq X \subseteq \overline{P}(X) \subseteq \overline{R}(X)$.

2- $\underline{R}\ \underline{P}(X) = \underline{R}\left(X \cap \underline{R}\overline{R}(X)\right) = \underline{R}(X) \cap \underline{R}\left(\underline{R}\overline{R}(X)\right)$,

since $\underline{R}(X) \subseteq \underline{R}\overline{R}(X) \Longrightarrow \underline{R}(X) \subseteq \underline{R}\left(\underline{R}\overline{R}X\right), \underline{R}\ \underline{P}(X) = \underline{R}(X)$.

On the other hand

$\underline{P}\ \underline{R}(X) = \underline{R}(X) \cap \underline{R}\overline{R}\left(\underline{R}(X)\right) = \underline{R}(X) \cap \quad \underline{R}\left(\overline{R}\underline{R}(X)\right)$. Since

$\underline{R}(X) \subseteq \overline{R}\underline{R}(X)$ and $\underline{R}(X) = \underline{RP}(X)$, then $\underline{P}\left(\underline{R}(X)\right) = \underline{R}(X)$.

3- Firstly $\overline{RP}(X) = \overline{R}\left[X \cup \overline{R}\underline{R}(X)\right] = \overline{R}(X) \cup \overline{R}\left(\overline{R}\underline{R}(X)\right)$,

since $\overline{R}\underline{R}(X) \subseteq \overline{R}(X)$ & $\overline{R}(X) = \overline{RR}(X)$, then $\overline{RP}(X) = \overline{R}(X)$. Also,

On the other hand

$\overline{P}\,\overline{R}(X) = \overline{R}(X) \cup \overline{R}\underline{R}\left(\overline{R}(X)\right) = \overline{R}(X) \cup \overline{R}\left(\underline{R}\overline{R}(X)\right)$, since

$\underline{R}\overline{R}(X) \subseteq \overline{R}(X)$ & $\overline{R}(X) = \overline{RR}(X)$, $\overline{P}\,\overline{R}(X) = \overline{R}(X)$.

Proposition 1.6

$\underline{P}(X - Y) \subseteq \underline{P}(X) - \underline{P}(Y)$.

Proof

Since $(X - Y) = (X \cap Y^C)$, then: $\underline{P}(X - Y) = \underline{P}(X \cap Y^C) \subseteq \underline{P}(X) \cap \underline{P}(Y^C)$.

By using that $\left[\underline{P}(X^C) = \left(\overline{P}(X^C)\right)^C\right]$, we have

$\underline{P}(X - Y) \subseteq \underline{P}(X) \cap \left(\overline{P}(Y^C)\right)^C = \underline{P}(X) - \overline{P}(Y^C) \subseteq \underline{P}(X) - \overline{P}(Y)$. Hence $\underline{P}(X - Y) \subseteq \underline{P}(X) - \underline{P}(Y)$.

The following example show that $\overline{P}(X - Y) \supseteq \overline{P}(X) - \overline{P}(Y)$ in general is not true.

Example 1.8

Let $U = \{a, b, c, d\}$, $X \subseteq U$, where $X = \{b, c\}$,

$R = \{(a, a), (b, a), (b, b), (b, c), (c, c), (d, b), (d, c), (d, d)\}$,

$S = \{\{a\}, \{a, b, c\}, \{c\}, \{b, c, d\}\}$,

$\beta = \{U, \Phi, \{a\}, \{a, b, c\}, \{c\}, \{b, c, d\}, \{b, c\}\}$, $T = \{U, \Phi, \{a\}, \{a, b, c\}, \{c\}, \{b, c, d\}, \{b, c\}, \{a, c\}\}$, $T^C = \{U, \Phi, \{b, c, d\}, \{d\}, \{a, b, d\}, \{a\}, \{a, d\}, \{b, d\}\}$.

Now X is $P - open$ if $X \subseteq \overset{\circ}{\overline{X}}$,

$PO(U) =$
$\{U, \Phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}\}, \{a, c, d\}, \{b, c, d\},$
$PC(U) =$
$\{U, \Phi, \{b, c, d\}, \{a, b, d\}, \{b, d\}, \{a, d\}, \{a, b\}, \{d\}, \{b\}, \{a\}\}$, if $X = \{a, c\}, Y = \{a, d\}$ then $X - Y = \{a\} \Longrightarrow \underline{R}(X - Y) = \{a\}$,
$\overline{P}(X - Y) = \{a\}$, $\underline{R}(X) = \{a, c\}$, $\overline{R}\underline{R}(X) = U \Longrightarrow \overline{P}(X) = U$ &
$\underline{R}(Y) = \{a\}$, $\overline{R}\underline{R}(Y) = \{a\} \Longrightarrow \overline{P}(Y) = \{a, d\}$, $\overline{P}(X) - \overline{P}(Y) = \{b, c\} \Longrightarrow \overline{P}(X - Y) \subseteq \overline{P}(X) - \overline{P}(Y)$.

In general, Properties $\overline{P}(X) \neq \underline{P}\overline{P}(X)$ & $\underline{P}(X) \neq \overline{P}\left(\underline{P}(X)\right)$ cannot be applied for lower and upper approximations. The following example illustrates this fact.

Example 1.10

Let $U = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (d, d)\}$, $S = \{\{d\}, \{a, b\}\}$, $\beta = \{U, \Phi, \{d\}, \{a, b\}\}$,
$T = \{U, \Phi, \{d\}, \{a, b\}, \{a, b, d\}\}$,
$T^C = \{U, \Phi, \{c, d\}, \{c\}, \{a, b, c\}\}$, and hence
$PO(U) = \begin{Bmatrix} U, \Phi, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \\ \{b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\} \end{Bmatrix}$, $PC(U) = \begin{Bmatrix} U, \Phi, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \\ \{a, b, c\}, \{b, c\}, \{a, c\}, \{c\}, \{a\} \end{Bmatrix}$.

If $X = \{b, c\}$, $\underline{P}(X) = \{b\}$, $\underline{P}\left(\underline{P}(X)\right) = \{b\}$ & $\overline{P}\left(\underline{P}(X)\right) = \{b, c\} \Longrightarrow \underline{P}\left(\underline{P}(X)\right) = \underline{P}(X) \neq \overline{P}\left(\underline{P}(X)\right)$. Also $\overline{P}(X) = \{b, c\} \Longrightarrow \overline{P}\left(\overline{P}(X)\right)\{b, c\}$ & $\underline{P}\left(\overline{P}(X)\right) = \{b\} \Longrightarrow \overline{P}\left(\overline{P}(X)\right) = \overline{P}(X) \neq \underline{P}\left(\overline{P}(X)\right)$.

Either $\underline{P}(X \cap Y) \neq \underline{P}(X) \cap \underline{P}(Y)$, &$\overline{P}(X \cup Y) \neq \overline{P}(X) \cup \overline{P}(Y)$.

2- Prepositive, prenegative and preboundary regions

Definition 2.1

Let $K = (U, \mathbb{R})$, be a general knowledge base and, $X \subseteq U$. Then the preboundary region of X is denoted by $BN_P(X)$, and defined by

$BN_P(X) = \overline{P}(X) - \underline{P}(X)$. The prepositive region of X is denoted by

$Pos_P(X)$, and defined by $POs_P(X) = \underline{P}(X)$. Also the prenegative region of X is denoted by $NEG_P(X)$, and defined by $NEG_P(X) = U - \overline{P}(X)$.

Definition 2.2

Let $K = (U, \mathbb{R})$, be a general knowledge base and, $X \subseteq U$. Then we

must define the external preboundary, and the internal preboundary

regions of X as,

$BN_{P_{EXT}}(X) = \overline{R}(X) - \underline{P}(X)$, $BN_{P_{INT}}(X) = \overline{P}(X) - \underline{R}(X)$,

respectively as in the following figure (1):

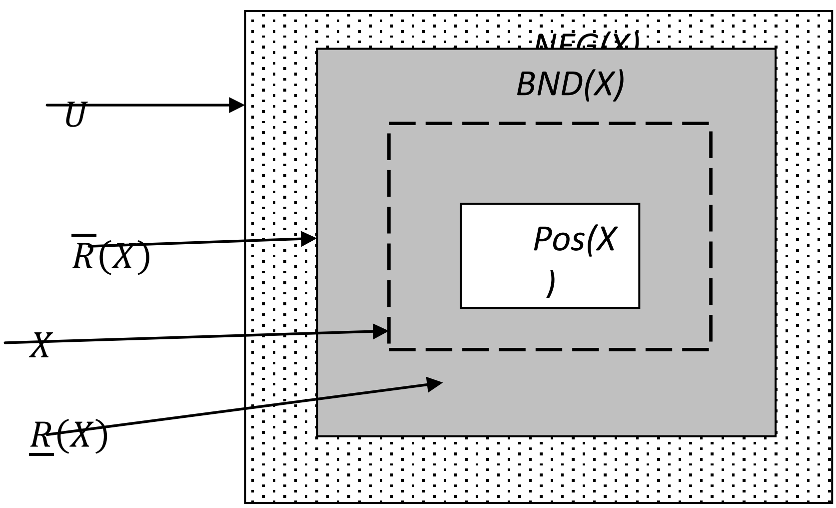
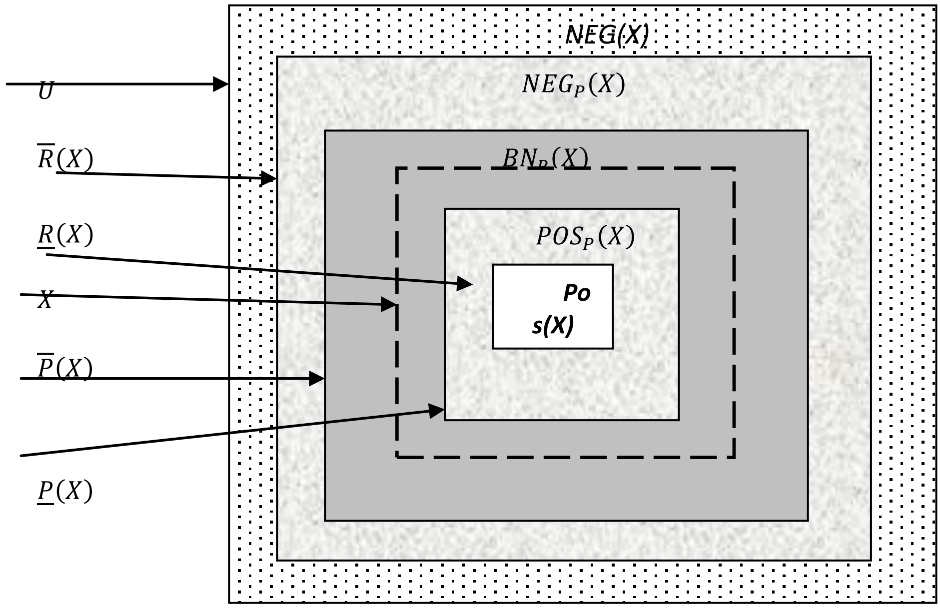


Figure (1)

Figure (2) illustrates the relations between $BN(X)$ and $BN_P(X)$ for a subset $X \subseteq U$, in a knowledge approximation space $(U, \mathbb{R})$.



Proposition 2.1

Let $(U, \mathbb{R})$, be a general knowledge base and $R \in \mathbb{R}$, and let $X \subseteq U$.

Then the following statements are true

1- $BN_P(X) \subseteq B(X)$.

2- $BN_{P_{EXT}}(X) \subseteq B(X)$.

3- $BN_P(X) \subseteq BN_{P_{INT}}(X)$.

Proof

1- $BN_P(X) = \overline{P}(X) - \underline{P}(X) \subseteq \overline{P}(X) - \underline{R}(X) \subseteq \overline{R}(X) - \underline{R}(X) = BN(X)$.

(2) and (3) are obvious.

Example 2.1

Let $U = \{a, b, c, d\}$, and R be a general relation where: $R = \{(a, a), (a, c), (c, b), (c, c), (c, d), (d, b), (d, d)\}$, then

$S = \{\{a,c\},\{b,c,d\},\{b,d\}\}$,
$\beta = \{U,\Phi,\{a,c\},\{b,c,d\},\{b,d\},\{c\}\}$,
$T = \{U,\Phi,\{a,c\},\{b,c,d\},\{b,d\},\{c\}\}$,
$T^C = \{U,\Phi,\{b,d\},\{a\},\{a,c\},\{a,b,d\}\}$, $PO(U) = \left\{\begin{matrix} U,\Phi,\{b,d\},\{b,c\},\{a,c\},\{b\}, \\ \{c\},\{d\},\{c,d\},\{a,b,c\},\{b,c,d\} \end{matrix}\right\}$,
$PC(U) = \left\{\begin{matrix} U,\Phi,\{a,c\},\{a,d\},\{b,d\},\{a,c,d\}, \\ \{a,b,d\},\{a,b,c\},\{a,b\},\{d\},\{a\} \end{matrix}\right\}$. If $X = \{a,c,d\}$ then $\underline{R}(X) = \{a,c\} \Rightarrow \overline{R}\underline{R}(X) = U$, $\overline{R}(X) = U \Longrightarrow \underline{R}\overline{R}(X) = U \Longrightarrow \overline{P}(X) = U$, $\underline{P}(X) = \{a,c,d\}$, $BN_P(X) = \{b\}$, $B(X) = \{b,d\}$, $BN_{PEX}(X) = \{b\}$, $BN_{P_{INT}}(X) = \{b,d\} \Longrightarrow BN_P(X) \subseteq B(X)$, $BN_{P_{EXT}}(X) \subseteq B(X)$, $BN_P(X) \subseteq BN_{P_{INT}}(X)$.

Proposition 2.2

Let $K = (U,\mathbb{R})$ be a general knowledge base, and $R \in \mathbb{R}$, $X \in U$ then

1- $NEG(X) \subseteq NEG_P(X)$.
2- $NEG_P(X \cup Y) \subseteq NEG_P(X) \cup NEG_P(Y)$.
3- $NEG_P(X \cap Y) \supseteq NEG_P(X) \cap NEG_P(Y)$.

Proof

1- Since $\overline{P}(X) \subseteq \overline{R}(X) \Longrightarrow U - \overline{R}(X) \subseteq U - \overline{P}(X)$ and
$NEG(X) = U - \overline{R}(X) \subseteq U - \overline{P}(X) = P - NEG(X)$, then
$NEG(X) \subseteq P - NEG(X)$.

2- $P - NEG(X \cup Y) \subseteq P - NEG(X) \cup P - NEG(Y)$
$\overline{P}(X \cup Y) \supseteq \overline{P}(X) \cup \overline{P}(Y) \Longrightarrow U - \overline{P}(X \cup Y) \subseteq U - \left(\overline{P}(X) \cup \overline{P}(Y)\right)$
$\Longrightarrow P - NEG(X \cup Y) = U - \overline{P}(X \cup Y) \subseteq U - \left(\overline{P}(X) \cup \overline{P}(Y)\right)$
$= \left(\overline{P}(X) \cup \overline{P}(Y)\right)^C = \left(\overline{P}(X)\right)^C \cap \left(\overline{P}(Y)\right)^C$
$= \left(U - \overline{P}(X)\right) \cap \left(U - \overline{P}(Y)\right)$

$= P - NEG(X) \cap P - NEG(Y) \quad \subseteq P - NEG(X) \cup P - NEG(Y)$.

3- Since $\overline{P}(X \cap Y) \subseteq \overline{P}(X) \cap \overline{P}(Y) \Longrightarrow$

$U - \left(\overline{P}(X) \cap \overline{P}(Y)\right) \subseteq U - \overline{P}(X \cap Y)$ &

$P - NEG(X \cap Y) = U - \overline{P}(X \cap Y) \supseteq U - \left(\overline{P}(X) \cap \overline{P}(Y)\right)$

$\supseteq U - \left(\overline{P}(X) \cup \overline{P}(Y)\right)$

$= \left(U - \overline{P}(X)\right) \cap \left(U - \overline{P}(Y)\right) = P - NEG(X) \cup P - NEG(Y)$.

Definition 2.3

Let $(U, \mathbb{R})$ be a general knowledge base and X is a finite nonempty subset of U. Then the pre accuracy of X is denoted by $\eta_P(X)$ and defined by $\eta_P(X) = \frac{|\underline{P}(X)|}{|\overline{P}(X)|}$, where $\left|\overline{P}(X)\right| \neq \Phi$.

Proposition 2.3

Let $(U, \mathbb{R})$ be a general knowledge base. If X be a finite nonempty subset of U. Then

$\eta(X) \leq \eta_P(X)$, where $\eta(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|}$ is the accuracy of U.

Proof Obvious.

Example 2.3

Using the same general knowledge base as in (Example 2.1)

If $X = \{a, c, d\}$, then $\eta(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|} = \frac{1}{2}$, $\eta_P(X) = \frac{|\underline{P}(X)|}{|\overline{P}(X)|} = \frac{3}{4}$.

Thus $\eta(X) < \eta_P(X)$.

3- Preapproximations and membership relation

The concept of preapproximation of sets leads to a new concept of membership relations. These membership relations must be related to knowledge because definition of a set in our approach is associated with knowledge. In case of Pawlak's approximation space there are two membership relations, but in our approach there are four membership relations which can be defined as follows:

Definition 3.1

Let $K = (U,\mathbb{R})$ be a general knowledge base, $X \subseteq U$ and $R \in \mathbb{R}$. For any $x \in U$ then:-

(1) $x \underline{\in}_R X \ iff \ x \in \underline{R}X$.

(2) $x \overline{\in}^R X \ iff \ x \in \overline{R}X$.

(3) $x \underline{\in}_P X \ iff \ x \in \underline{P}X$.

(4) $x \overline{\in}^P X \ iff \ x \in \overline{P}X$. Where

(i) $x \in \underline{R}X$ Or $x \in \underline{P}X$ means that x certainly belongs to X.

(ii) $x \in \overline{P}X$ means that x possibly belongs to X.

(iii) $x \in \overline{R}X - \overline{P}X$ means that x definitely does not belong to X.

So, the concepts of prelower and preupper approximations enlarge the number of possible membership relations.The advantage of the new membership relation is to help the decision maker to a variety of choices instead of two in Pawlak rough set model and three in general rough set models. If R is an equivalence relation, then all four membership relations coincide. The next Proposition illustrates the properties of pre membership relation and the difference from the usual one.

Proposition 3.1

Let $K = (U,\mathbb{R})$ be a general knowledge base, $X \subseteq U$ and $R \in \mathbb{R}$.

$x \in U$, then:-

(1) $x \underline{\in}_R X \Rightarrow x \underline{\in}_P X \Rightarrow x \in X \Rightarrow x \overline{\in}^P X \Rightarrow x \overline{\in}^R X.$

(2) $x \overline{\in}^R (X \cup Y)\ iff\ x \overline{\in}^R X\ or x \overline{\in}^R Y$.

(3) $x \overline{\in}^P X\ or\ x \overline{\in}^P Y \Rightarrow x \overline{\in}^P (X \cup Y).$

(4) $x \overline{\in}^R (X \cap Y) \Rightarrow x \overline{\in}^R X\ \&\ x \overline{\in}^R Y.$

(5) $x \overline{\in}^P (X \cap Y) \Rightarrow x \overline{\in}^P X\ \&\ \overline{\in}^P Y.$

(6) $x \underline{\in}_R X\ or\ x \underline{\in}_R Y \Rightarrow x \underline{\in}_R (X \cup Y).$

(7) $x \underline{\in}_P X\ or\ x \underline{\in}_P Y \Rightarrow x \underline{\in}_P (X \cup Y).$

(8) $x \underline{\in}_R (X \cap Y)\ iff x \underline{\in}_R X\ \&\ x \underline{\in}_R Y.$

(9) $x \underline{\in}_P (X \cap Y) \Rightarrow x \underline{\in}_P X\ \&\ x \underline{\in}_P Y.$

Proof

It follows immediately from (Proposition 1.5) the following

example shows that the converse of the previous proposition is not in general true

Example 3.1

Let $U = \{a, b, d, e\}$ and R be a general relation defined as

$R = \{(a,a), (a,c), (c,b), (c,c), (c,d), (e,a), (e,e)\}$,
$S = \{\{a,c\}, \{b,c,d\}, \{a,e\}\}$,
$\beta = \{U, \Phi, \{a,c\}, \{b,c,d\}, \{a,e\}, \{c\}, \{a\}\}$,

$T = \begin{Bmatrix} U, \Phi, \{c\}, \{a\}, \{a,c\}, \{b,c,d\}, \\ \{a,e\}, \{a,c,e\}, \{a,b,c,d\} \end{Bmatrix}$, $T^C =$
$\begin{Bmatrix} U, \Phi, \{e\}, \{b,d,e\}, \{a,e\}, \{b,c,d\}, \\ \{a,b,d,e\}, \{b,c,d,e\}, \{b,d\}, \end{Bmatrix}$

(1) Let $X = \{c,d,e\}$, $\underline{R}(X) = \{c\}$, $\overline{R}\left(\underline{R}(X)\right) = \{b,c,d\}$,

$\overline{R}(X) = \{b,c,d,e\}$, $\underline{R}\left(\overline{R}(X)\right) = \{b,c,d\} \Rightarrow$

$\underline{P}(X) = \{c,d,e\}$, $\overline{P}X = \{b,c,d,e\}$ so $d \underline{\in}_P X$
but $d \underline{\notin}_R X$.

Let $X = \{a,b,e\}$, $\underline{R}(X) = \{a,e\}$, $\overline{R}(X) = \{a,b,d,e\}$,

$\overline{R}\left(\underline{R}(X)\right) = \{a, e\} \Longrightarrow \overline{P}X = \{a, b, e\}$, so $d\ \overline{\in}^{R}\ X$ but $d\ \overline{\notin}^{P}\ X$.

(2) Let $X = \{b, d\}$, $= \{c, e\}$ $\Longrightarrow X \cup Y = \{b, c, d, e\}$, so $\overline{P}X = \{b, d\}$,

$\overline{P}Y = \{b, c, d, e\}$ & $\overline{P}(X \cup Y) = \{b, c, d, e\}$, so $e \in \overline{P}(X \cup Y)$ but

$e \notin \overline{P}X$ & $e \in \overline{P}Y$. i.e. $e\ \overline{\in}^{P}\ (X \cup Y)$ but $e\ \overline{\notin}^{P}\ X$.

(3) Let $X = \{b, d\}$, $Y = \{a, c\}$,so $\overline{P}X = \{b, d\}$ & $\overline{P}Y = U$,hence $b \in \overline{P}X$

& $b \in \overline{P}Y$, but $X \cap Y = \Phi$, so $\overline{P}(X_1 \cap X_2) = \Phi$, and $b \notin \overline{P}(X \cap Y)$.

(4) Let $X = \{c, e\}$, $Y = \{b, e\}$ so $\underline{P}X = \{c\}$ & $\underline{P}Y = \Phi$. Also $X \cup Y = \{b, c, e\}$, hence $\underline{P}(X \cup Y) = \{b, c\}$.Therefore

$b \in \underline{P}(X \cup Y)$,but $b \notin \underline{P}X$, $b \notin \underline{P}Y$ .i.e. $b\ \underline{\in}_P\ (X \cup Y)$but $b\ \underline{\notin}_P\ X$,

$b\ \underline{\notin}_P\ Y$.

(5) Let $X = \{a, c, d\}$, $Y = \{a, b, d, e\}$, so $\underline{P}X = \{a, c, d\}$, $\underline{P}Y = \{a, e\}$.

$X \cap Y = \{a, d\}$, hence $\underline{P}(X \cap Y) = \{a\}$.

So $e \notin \underline{P}X$ & $e \in \underline{P}Y$, but $e \notin \underline{P}(X \cap Y)$. i.e. $x\ \underline{\in}_P\ X$ & $x\ \underline{\in}_P\ Y$ dose not in general imply $x\ \underline{\in}_P\ (X \cap Y)$.

4- Prerough membership function:

Original rough membership function is defined using equivalence classes. It was extended to topological spaces [3,7,11,12], namely, If T is a topology on a finite set X, where its base is β, then the rough membership function is $\mu_X^P(x) = \frac{|\{\cap B_x\} \cap X|}{|\{\cap B_x\}|}$, $x \in X$, where B_x is any member of β containing x & $X \subseteq U$.

We introduce the following definition for a prerough membership function to express $BN_P(X), PO_P(X), NEG_P(X)$, for a subset $X \subseteq U$.

Definition 4.1

Let $K = (U, \mathbb{R})$ be a general knowledge base, and $X \in U$**,** then the

preroughh membership function on U is:

$\mu_X^P(x): U \longrightarrow [0,1]$, and it is defined by

$$\mu_X^P(x) = \begin{cases} 1, & if\ 1 \in B_x(X) \\ minB_x(X), & otherwise \end{cases} \quad where$$

$$B_x(X) = \left\{\frac{|B \cap X|}{|B|} : B\ is\ a\ P - open\ set, x \in B\right\}.$$

Definition 4.2

The rough premembership function may be used to define pre

approximations and preboundary region of a set, as shown below:

$\underline{P}(X) = \{x \in U: \mu_X^P(x) = 1\}$, $\overline{P}(X) = \{x \in U: \mu_X^P(x) > 0\}$,

$BN_P(X) = \{x \in U: 0 < \mu_X^P < 1\}$.

It can be shown that the membership function has the following properties:

Proposition 4.1

Let $(U, \mathbb{R})$ be a general knowledge base, and $X \subseteq U$ then:

1- $x \in \underline{P}(X) \Leftrightarrow \mu_X^P(x) = 1$.

2- $x \in BN_P(X) \Leftrightarrow 0 < \mu_X^P(x) < 1$.

3- $x \in U - \overline{P}(X) = EXT_P(X) \Leftrightarrow \mu_X^P(x) = 0$.

Proof

1- $x \in \underline{P}(X) \Leftrightarrow \exists\ A \in PO(X)\ s.t. x \in A \subseteq X$

$$\Leftrightarrow \exists\ A \in PO(X), x \in A\ s.t.\ \frac{|A \cap X|}{|A|} = 1$$

$\Leftrightarrow \mu_X^P(x) = 1$.

2- $x \in BN_P(X) \Leftrightarrow\ \forall\ A \in PO(X), x \in A$ we have $A \cap X \neq \Phi$ &

$A \cap (U - X) \neq \Phi \Leftrightarrow \forall\ A \in PO(X), x \in A$, we have

$0 < |A \cap X| < |A| \Leftrightarrow 0 < \mu_X^P(x) < 1$.

3- $x \in EXT_P(X) \Leftrightarrow \exists A \in PO(X)\ s.t.\ x \in A \subseteq (U - X)$
$\Leftrightarrow \exists A \in PO(X), x \in A\ s.t.\ A \cap X = \Phi$
$\Leftrightarrow \exists A \in PO(X), x \in A\ s.t.$
$\Leftrightarrow \exists A \in PO(X), x \in A\ s.t.$ $\frac{|A \cap X|}{|A|}$
$= 0 \Leftrightarrow \mu_X^P(x) = 0.$

Definition 4.3

Let $(U, \mathbb{R})$ be a general knowledge base, $X \subseteq U$. Then we must define the three regions by:-

$POS_P(X) = \{x \in U: \mu_X^P(x) = 1\}$, $NEG_P(X) = \{x \in U: \mu_X^P(x) = 0\}$, $BN_P(X) = \{x \in U: \mu_X^P(x) < 1\}$.

Obviously, they use extreme values of $\mu_X^S(x)$, i.e. 0 and 1.

Example 4.1

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $R = \{(a,a),(b,b),(b,d),(c,c),(d,d),(e,a),(e,e)\}$, $S = \big\{\{a\},\{b,d\},\{c\},\{d\},\{a,e\}\big\}$, $\beta = \big\{U, \Phi, \{a\},\{b,d\},\{c\},\{a,e\},\{d\}\big\}$

$T = \left\{\begin{matrix} U,\Phi,\{a\},\{b,d\},\{c\},\{d\},\{a,e\},\{a,b,d\},\{a,c\},\{a,c,e\}, \\ \{c,d\},\{b,c,d\},\{a,b,c,d\},\{a,b,d,e\},\{a,c,d,e\},\{a,c,d\} \end{matrix}\right\}$,

$T^C(X) = \left\{\begin{matrix} U,\Phi,\{b,c,d,e\},\{a,b,d,e\},\{a,b,c,e\},\{b,d,e\},\{b,d\} \\ \{a,c,e\},\{b,e\},\{b,c,d\},\{a,b,e\},\{c,e\},\{e\},\{c\},\{b\},\{a,e\} \end{matrix}\right\}$,

$PO(U) = \left\{\begin{matrix} U,\Phi,\{a\},\{c\},\{d\},\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{b,d\}, \\ \{c,d\},\{a,b,d\},\{b,c,d\},\{c,d,e\},\{a,b,c,d\},\{a,b,d,e\} \end{matrix}\right\}$,

since $B_x \subseteq \beta$ & $if\ X = \{a,c\}$, then $\mu_X^P(a) = 1$, $\mu_X^P(b) = 0$, $\mu_X^P(c) = 1$, $\mu_X^P(d) = 0$, $\mu_X^P(e) = \frac{1}{2}$. Therefore

$\underline{P}(X) = \{x \in U: \mu_X^P(x) = 1\} = \{a,c\}$,

$\overline{P}(X) = \{x \in U: \mu_X^P(x) > 0\} = \{a,c\}$,

$BN_P(X) = \{x \in U: 0 < \mu_X^P < 1\} = \Phi$,

$POS_P(X) = \{x \in U: \mu_X^P(x) = 1\} = \{a,c\}$,

$NEG_P(X) = \{x \in U: \mu_X^P(x) = 0\} = \{a,d\}$.

Example 4.2

Let $U = \{a, b, d, e\}$ & $R = \{(c, c), (b, b), (b, d)\}$, $S = \{\{c\}, \{b, d\}\}$,

$\beta = \{U, \Phi, \{c\}, \{b, d\}\}$ & $T = \{U, \Phi, \{c\}, \{b, d\}, \{b, c, d\}\}$.

If $X = \{a, b, c\}$, then we get $\mu_X^P(a) = 1$, $\mu_X^P(b) = 1$, $\mu_X^P(c) = 1$,

$\mu_X^P(d) = 0$. From pre membership function, we get $\underline{P}(X) = \{a, b, c\}$, $\overline{P}(X) = \{a, b, c\}$, $BN_P(X) = \Phi$, $POS_P(X) = \{a, b, c\}$, $NEG_P(X) = \{d\}$.

Thus X is a $P - definable$ $(P - exact)$ set.

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