بعض الفئات من الدوال التحليلية بواسطة استخدام n-th جداء هادامارد

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ملخص :

في هذا البحث, نحن نشتق بعض الشروط الكافية لأجل بعض الفئات الفرعية من الدوال التحليلية بواسطة استخدام n-thجداء هادامارد.

ON SOME CLASSES OF ANALYTIC FUNCTIONS BY INVOLVING $n-$th HADAMARD PRODUCT

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Abstract:- In this paper, we derive some sufficient conditions for certain subclasses of analytic functions by involving $n-$ th Hadamard product.

Keywords:
analytic function, generalized operator, differential subordination.

1- Introduction

Let A denote the class of functions of the form:

$$f(z)=\sum_{k=2}^{\infty} a_k z^k, \qquad (1.1)$$

which are analytic in the open unit disc $U=\{z\in U:|z|<1\}$ and are normalized by the conditions $f(0)=f(0)-1=0$. A function $f;f'(0)\neq 0$, is said to be close-to-convex in U, if and only if, there is a starlike function h (not necessarily normalized) such that

$$\Re\{\frac{zf'(z)}{h(z)}\} > 0, \quad z \in U.$$

For two function $f(z)$ and $g(z)$ given by

$$f(z) = \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=2}^{\infty} b_k z^k,$$

(1.2)

Their Hadamard product is defined by

$$(f * g)(z) = \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let f and g be analytic in U. We say that f is subordinate to g in U, written as

$f(z) \prec g(z)$ in U, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subset g(U)$.

And several functions $f_1(z), f_2(z), ... f_n(z) \in A$, by $n-$th Hadamard product is defined as follows

$$(f_1 * f_2 ... * f_n)(z) = \sum_{k=2}^{\infty} a_{1k} a_{2k} ... a_{nk} z^k.$$

We recall here a general Hurwitz-Lerch Zeta function $\phi(z, s, b)$ defined by: (see, for example [cf., e.g.,[10]], [11]).

$$\phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

$$(b \in C \setminus \{\bar{Z}_0\}; s \in C, when\ |z| < 1, \Re(s) > 1\ when\ |z| = 1)$$

Where, as usual, $\bar{Z}_0 = Z \setminus \{N\}, Z = \{\pm 1, \pm 2, ...\}; N = \{1, 2, ...\}$. We define by Hadamard product $n-$th order as follows:

$$\psi(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) = \underbrace{\phi(z, s_1, b_1) * \phi(z, s_2, b_2) * ... \phi(z, s_n, b_n)}_{n-times}.$$

Now we de ne the linear operator $I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z): A^n \to A$, as follows:

$$I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z) = J(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) * f(z), \quad z \in U$$

where for convenience

$$J(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) = (1+b_1)^{s_1} ... (1+b_n)^{s_n} [\psi(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) - b_1^{-s_1} .. b_n^{-s_1}].$$

It is easy to observe we get

$$I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z) = z + \sum_{k=0}^{\infty} \prod_{i=1}^{n} \left(\frac{1+b_i}{k+b_i} \right)^{s_i} a_k z^k$$

(1.3)

where $b_1, b_2, ..., b_n \in C \setminus \{\bar{Z}_0\}; s_1, s_2, ..., s_n \in C$ and $z \in U$.

Also we define the integral operator as follows:

$$J(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) * J^{(-1)}(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) = \frac{z}{(1-z)^{1+\lambda}},$$

we have

$$L_\lambda(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z) = J^{(-1)}(z, s_1, s_2, ..., s_n, b_1, b_2, ..., b_n) * f(z)$$

$$= z + \sum_{k=2}^{\infty} \prod_{i=1}^{n} \left(\frac{1+b_i}{k+b_i} \right)^{s_i} \frac{(\lambda+1)_{k-1}}{(k-1)!} a_k z^k,$$

(1.4)

where $b_1, b_2, ..., b_n \in C \setminus \{\bar{Z}_0\}; s_1, s_2, ..., s_n \in C$ and $\lambda > -1$.

For $I(s_1,0,...,0,b_1,b_2,...,b_n)(f)(z)$ were introduced by Srivastava and Attiya [9] (see also Raducanu and Srivastava [7], Liu [2] and Prajapat et.al.[6]). And for $L_\lambda(s_1,0,...,0,b_1,b_2,...,b_n)(f)(z)$ where $s_1,\lambda \in N_0 = N \cup \{0\}$ were introduced by Al-Shaqsi and M.Darus [1], and for $L_\lambda(0,0,...,0,b_1,b_2,...,b_n)(f)(z)$ is the differential operator defined by Ruscheweyh[8].

For a function $f(z) \in A$, we say that $f \in R(s_1,s_2,...,s_n,b_1,b_2,...,b_n,\alpha,\delta)$ if it satisfies

$$\left|(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}\right| < \delta$$

for some real $\alpha(-\pi \le \alpha \le \pi), \delta > \sqrt{2(1-\cos\alpha)}, b_1,b_2,...,b_n \in C\setminus\{\bar{Z}_0\}; s_1,s_2,...,s_n \in C$, and for some $g \in A$. Furthermore, a function $f(z) \in A$ is said to be in the class $f \in R*(0,0,...0,b_1,b_2,...,b_n,\lambda,\alpha,\delta)$ if it satisfies

$$\left|(L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}\right| < \delta$$

for some real $\alpha(-\pi \le \alpha \le \pi), \delta > \sqrt{2(1-\cos\alpha)}, b_1,b_2,...,b_n \in C\setminus\{\bar{Z}_0\}; s_1,s_2,...,s_n \in C, \lambda > -1.$

For $f \in R(0,0,...0,b_1,b_2,...,b_n,\alpha,\delta)$ were introduced by S.Owa,Y. Polatoglu at.al.[4].

In the present study, we apply a method based on the differential subordination

to obtain sufficient conditions for certain subclasses of analytic function by involving $n-$th Hadamard product.

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \prec q(z)$$

and

$$(L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \prec q(z).$$

2- Preliminaries

We shall need following definition and lemmas to prove our results.

Definition 2.1

A function $L(z,t); z \in U$ and $t \geq 0$ is said to be a subordination chain if $L(:,t)$ is analytic and univalent in U for all $t \geq 0, L(z,:)$ is continuously differentiable on $[0,1)$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 2.2 [5]

The function $L(z,t): U \times [0,1) \to C$ (C is the set of complex numbers), of the

form $L(z,t) = a_1(t)z + ...$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t\to\infty}|a_1(t)| = \infty$, is said to be a subordination chain if and only if $\Re\{\frac{z\ \partial L/\partial z}{\partial L/\partial t}\} > 0,$

for all $z \in U$

and $t \geq 0$.

Lemma 2.3 [3]

Let $p(z)$ be analytic in U and let q be analytic and univalent in $\bar{U}$ except for points ζ_0 such that $\lim_{\zeta_0\to\infty} p(z) = \infty, p(z) = 1,$ with $p(0) = q(0).$ If $p \not\prec q$ in U, then there is a point $z_0 \in U$ and $\zeta_0 \in \partial U$, (boundary of U) such that $p(|z| < |z_0|) \subset q(E), p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(z_0)$ for some $m \geq 1$.

3- Main Result

Theorem 3.1

Let $\Re\beta \geq 0$ be a complex number and $\alpha (0 \leq \alpha \leq \frac{\pi}{2})$. Let q be univalent function

such that either $\frac{1}{q(z)}$ is convex in U. If

analytic, satisfies the differential subordination

$$\beta - \frac{e^{i\alpha}}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec \beta - \frac{e^{i\alpha}}{q(z)} + \frac{zq'(z)}{q^2(z)} \quad (3.1)$$

then

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof:

Let h a function define

$$h(z) = \beta - \frac{e^{i\alpha}}{q(z)} + \frac{zq'(z)}{q^2(z)} \quad (3.2)$$

Differentiating (3.2) and simplifying a little, we have

$$\frac{zh'(z)}{F(z)} = e^{i\alpha} + \frac{zF'(z)}{F(z)} \quad (3.3)$$

where $F(z) = \frac{zq'(z)}{q^2(z)}$, then we obtain $\Re\{\frac{zh'(z)}{F(z)}\} \geq 0, z \in U$.

Thus $h(z)$ is close-to-convex and hence univalent in U. We need to show that

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \prec q(z).$$

Suppose to the contrary that

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \not\prec q(z).$$

Then by Lemma 2.2, there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(z_0); m \geq 1$. Than

$$\beta - \frac{e^{i\alpha}}{p(z_0)} + \frac{zp'(z_0)}{p^2(z_0)} = \beta - \frac{e^{i\alpha}}{q(z_0)} + \frac{zm\zeta q'(z_0)}{q^2(z_0)}. \qquad (3.4)$$

Consider a function $L(z,t)$ is analytic in U for all $t \geq 0$ and is continuously differentiable on $[0,1)$ for all $z \in U$, as follows:

$$L(z,t) = \beta - \frac{e^{i\alpha}}{q(z)} + (1+t)\frac{z\zeta q'(z)}{q^2(z)}, \qquad (3.5)$$

we have $a_1(t) = (\frac{\partial L(z,t)}{\partial z})_{(0,1)} = q'(0)(e^{i\alpha} + 1 + t)$. In view of the condition $0 \leq \cos\alpha \leq 1$ and $t \geq 0$. Also, as q is univalent in U, so, $q'(0) \neq 0$. Therefore, it follows that $a_1(0) \neq 0$ and $\lim_{t\to\infty}|a_1(t)| = \infty$. A simple calculation yields $\{\frac{z\,\partial L/\partial z}{\partial L/\partial t}\} = e^{i\alpha} + (1+t)\frac{zF'(z)}{F(z)}$. Then we get $\Re\{\frac{z\,\partial L/\partial z}{\partial L/\partial t}\} > 0$ in view of given conditions. Hence, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. From (3.5), we have $L(0,t) = h(z)$, thus we deduce that

$L(\zeta_0,t)\notin h(U)$ for $|\zeta_0|=1$ and $t\geq 0$. In view of (3.4) and (3.5), we can write

$\beta-\frac{e^{i\alpha}}{p(z_0)}+\frac{zp'(z_0)}{p^2(z_0)}=L(\zeta_0,m-1)\notin h(U)$ where $z_0\in U,|\zeta_0|=1$ and $m\geq 1$

which is contradiction to (3.1). Hence

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))'-e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}\prec q(z).$$

Using a similar method, we can prove the following theorem.

Theorem 3.2

Let $\alpha(0\leq\alpha\leq\frac{\pi}{2})$. Let q be univalent function such that either

$\frac{1}{q(z)}$ is convex in U. If

$$p(z)=(L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))'-e^{i\alpha}\frac{L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}$$

analytic, satisfies the differential subordination

$1-\frac{e^{i\alpha}}{p(z)}+\frac{zp'(z)}{p^2(z)}\prec 1-\frac{e^{i\alpha}}{q(z)}+\frac{zq'(z)}{q^2(z)}$, then

$$(L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))'-e^{i\alpha}\frac{L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}\prec q(z)$$

and $q(z)$ is the best dominant.

By taking $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$ in Theorem 3.1, we have the following

Corollary 3.3

Let $\Re\beta \geq 0$ be a complex number and A and B $-1 \leq B < A \leq 1$ and $\alpha(0 \leq \alpha \leq \frac{\pi}{2})$. Let q be univalent function such that either $\frac{1}{q(z)}$ is convex in U. If

$$p(z) = (I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z))' - e^{i\alpha} \frac{I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(g)(z)}{z}$$

analytic, satisfies the differential subordination

$$\beta - \frac{e^{i\alpha}}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec \beta - e^{i\alpha} \frac{1+Bz}{1+Az} + \frac{(A-B)z}{(1+Az)^2},\ \text{then}$$

$$(I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z))' - e^{i\alpha} \frac{I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(g)(z)}{z} \prec \frac{1+Az}{1+Bz}.$$

By taking $\beta = 0, B = -1, A = 1$ in Theorem 3.1, we have the following:

Corollary 3.4

Let $\alpha(0 \leq \alpha \leq \pi/2)$ Let q be univalent function such that either

$\frac{1}{q(z)}$ is convex in U.

$$p(z) = (I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(f)(z))' - e^{i\alpha} \frac{I(s_1, s_2, ..., s_n, b_1, b_2, ..., b_n)(g)(z)}{z}$$

analytic, satisfies the differential subordination

$$\beta - \frac{e^{i\alpha}}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec \beta - e^{i\alpha} \frac{1-z}{1+z} + \frac{2z}{(1+z)^2},\ \text{then}$$

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - e^{i\alpha}\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \prec \frac{1+z}{1-z}.$$

Taking $\alpha = \pi/2$ in Theorem 3.1, we have

Corollary 3.5

Let $\Re\beta \geq 0$ be a complex number. Let q be univalent function such that either

$\frac{1}{q(z)}$ is convex in U. If

$$p(z) = (I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - i\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}$$

analytic, satisfies the differential subordination

$$\beta - \frac{i}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec \beta - \frac{i}{q(z)} + \frac{zq'(z)}{q^2(z)}, \text{ then}$$

$$(I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))' - i\frac{I(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z} \prec q(z).$$

By taking $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ and $I(0,0,...,0,b_1,b_2,...,b_n)(f)(z)$ in Theorem 3.1, we have the following:

Corollary 3.6

Let $\Re\beta \geq 0$ be a complex number and $\alpha(0 \leq \alpha \leq \frac{\pi}{2})$. Let q be univalent function

such that either $\frac{1}{q(z)}$ is convex in U.

If $p(z)=f'(z)-e^{i\alpha}g(z)$ analytic, satisfies the differential subordination

$$\beta-\frac{e^{i\alpha}}{p(z)}+\frac{zp'(z)}{p^2(z)}\prec\beta-e^{i\alpha}\frac{1+Bz}{1+Az}+\frac{(A-B)z}{(1+Az)^2},\ \text{then}$$

$$f'(z)-e^{i\alpha}\frac{g(z)}{z}\prec\frac{1+Az}{1+Bz}$$

By taking $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ and $\alpha=0$ in Theorem 3.2, we have the following

Corollary 3.7

Let q be univalent function such that either

$\frac{1}{q(z)}$ is convex in U. If

$$p(z)=(L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))'-e^{i\alpha}\frac{L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}$$

analytic, satisfies the differential subordination

$$1-\frac{1}{p(z)}+\frac{zp'(z)}{p^2(z)}\prec 1-\frac{1+Bz}{1+Az}+\frac{(A-B)z}{(1+Az)^2},\ \text{then}$$

$$(L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(f)(z))'-\frac{L_\lambda(s_1,s_2,...,s_n,b_1,b_2,...,b_n)(g)(z)}{z}\prec\frac{1+Az}{1+Bz}.$$

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