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OSCILLATION CRITERION FOR SOME DIFFERENTIAL EQUATION OF SECOND ORDER WITH INTEGRABLE COEFICIENTS

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ABSTRACT

In this presented paper, we are concerned with the oscillatory behaviour of solutions of some nonlinear ordinary differential equations of second order. Primarily, we are interested in the study of equations with integrable coefficients. By using the Riccati substitution technique, many open problems are discussed and some new sufficient conditions for the oscillation are obtained. Some applications of the obtained results are also provided to show the feasibility of the new results.

Keywords: Oscillation; Nonlinear differential equations; Second Order; Integrable Coefficients.

الملخص

في هذه الورقة، نهتم بسلوك التذبذب لحلول بعض المعادلات التفاضلية العادية غير الخطية من الرتبة الثانية، تحديدا المعادلات ذات المعاملات القابلة للتكامل حيث تمت مناقشة عدد من القضايا باستخدام تعويض ريكاتي وتم الحصول على عدد من الشروط الكافية للتذبذب. تم تزويد البحث ببعض التطبيقات للنتائج المتحصل عليها لبيان جدوى النتائج الجديدة.

الكلمات المفتاحية : التذبذب، المعادلات التفاضلية غير الخطية، الرتبة الثانية، معاملات قابلة للتكامل.

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1. INTRODUCTIO In physical and engineering problems, questions related to the oscillation theory play an important role. As a result, there has been much activity concerning with oscillatory and asymptotic behavior of many different classes of differential equations, see for example [1, 3–10] and the references cited therein. This paper deals with the problem of the oscillation of all solutions of the following second order nonlinear differential equation

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)|x(t)|^{\gamma} sign x(t) = o, \quad t \geq t_0 \tag{1}$$

where q and r are continuous functions on the interval $[t_0, \infty\,)$, $t_0 \geq t$, $r(t)$ is a positive function on the real line $\Re$ and $\gamma > 0$. Our attention here is concentrated only to such solution $x(t)$ of the differential equation (1) which exists on some interval $[t_0, \infty\,), t_0 \geq 0$.

Definition 1. A solution $x(t)$ of the differential equation (1) is said to be nontrivial if $x(t) \neq 0$ for at least one t in the interval $[t_0, \infty\,), t_0 \geq 0$.

Definition 2. A nontrivial solution $x(t)$ of the differential equation (1) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty\,), t_0 \geq 0$, otherwise, it is said to be nonoscillatory.

Definition 3. Equation (1) is called oscillatory if all of its solutions are oscillatory, otherwise, it is said to be nonoscillatory.

Recently, Ahmed and Ali [2] provided some new criteria for the oscillation of equation (1) and their results are for the sublinear case, that is, for $0 < \gamma < 1$, and their results are not applicable to equation (1) with $\gamma > 0$. Also, one can see the papers of Elabbasy and Elzeiny [5], Grace and Lalli [7], Philos [13] and Yeh [16]. However, the main purpose of this paper is to establish some new criteria for the oscillation of equation (1) for the superlinear case, that is, for $\gamma > 1$, by using an averaging condition of the type introduced by Kamenev [8]. Therefore, results obtained here can be considered as a continuation of the work done by Ahmed and Ali [2]. As a contribution to this study we refer to the papers of Grace [6], Yeh [16] and Saad et al. [14].

2. MAIN RESULTS

In this section we state and prove some results of the oscillation of the equation (1). Some examples are also listed here to show the evidence of our own results.

Theorem 1

Suppose that

(O_1) $0 < A \leq r(t) \leq B, t \geq 0$, and $A, B \in R$

(O_2) $\dot{r}(t) > 0, t \geq 0$,

(O_3) $\liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds \geq -\lambda > -\infty \; ; \; \lambda > 0$,

(O_4) $\lim_{t\to\infty} sup \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds = \infty.$

Then the equation (1) is oscillatory when $\gamma > 1$.

Proof : Assume the contrary ; then there exists a solution $x(t)$ which may be assumed to be positive on $[T_1,\infty)$ for some $T_1 \geq t_0$. Dividing (1) through by $x^{\gamma}(t)$ and integrating from T_1 to t ,we obtain that

$$\frac{r(t)\dot{x}(t)}{x^{\gamma}(t)} + \gamma \int_{T_1}^{t} \frac{r(s)\dot{x}^2(s)}{x^{\gamma}} ds + \int_{T_1}^{t} q(s)ds = C, \quad C = \frac{r(T_1)\dot{x}(T_1)}{x^{\gamma}(T_1)} \tag{2}$$

Now, from the condition (O_1), since $A \leq r(t)$, then we have from (2),

$$\frac{r(t)\dot{x}(t)}{x^{\gamma}(t)} + \gamma A \int_{T_1}^{t} \left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds + \int_{T_1}^{t} q(s)ds \leq C, \quad \beta = \frac{(\gamma+1)}{2} \tag{3}$$

Integrating (3) once more from T_1 to t we obtain

$$\int_{T_1}^{t} r(s)\frac{\dot{x}(s)}{x^{\gamma}(s)} ds + \gamma A \int_{T_1}^{t}\int_{T_1}^{s} \left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds + \int_{T_1}^{t}\int_{T_1}^{s} q(u)duds \leq C(t-T_1) \tag{4}$$

For the first integral of (4), we have from the condition (O_2) that $r(t)$ is a non-decreasing on $t \geq T_1 \geq t_0$. So, by the Bonnet's theorem there exists $T_2 \in [T_1, t]$ such that

$$\int_{T_1}^{t} \frac{r(s)\dot{x}(s)}{x^{\gamma}(s)} ds = r(t)\int_{T_2}^{t} \frac{\dot{x}(s)}{x^{\gamma}(s)} ds = \frac{r(t)}{(1-\gamma)}\left[x^{-\gamma+1}(t) - x^{-\gamma+1}(T_2)\right]$$

By substituting in (4) we obtain that

$$\frac{r(t)x^{1-\gamma}(t)}{1-\gamma} + \gamma A \int_{T_1}^{t}\int_{T_1}^{s} \left(\dot{x}(u)\big/ x^{\beta}(u)\right)^2 duds + \int_{T_1}^{t}\int_{T_1}^{s} q(u)\,du\,ds \leq C(t-T_1) + \frac{r(t)x^{1-\gamma}(T_2)}{1-\gamma} \tag{5}$$

Since $0 < r(t) \leq B$ *and* $\gamma > 1$. Then, by the assumption that $x(t) > 0$, we have

$$\frac{r(t)x^{1-\gamma}(t)}{1-\gamma} \geq \frac{B x^{1-\gamma}(t)}{1-\gamma} \quad \text{and} \quad \frac{r(t)x^{1-\gamma}(T_2)}{1-\gamma} < 0 .$$

Hence (5) becomes

$$\frac{\beta}{(1-\gamma)} x^{1-\gamma}(t) + \gamma A \int_{T_1}^{t}\int_{T_1}^{s} \left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds + \int_{T_1}^{t}\int_{T_1}^{s} q(u)duds \leq C(t-T_1) \tag{6}$$

We distinguish three cases of the behaviour of $\dot{x}(t)$, namely,

(i) $\dot{x}(t)$ oscillatory on $[T_1,\infty)$,

(ii) $\dot{x}(t) > 0 \quad on[T_3,\infty)\ for\, some T_3 \geq T_1$,

(iii) $\dot{x}(t) < 0 \quad on[T_3,\infty)\ for\, some T_3 \geq T_1$,

and show that the assumption $x(t) > 0$ leads to a contradiction in each case. Suppose that $\dot{x}(t)$ oscillatory, then there exists a sequence $\{t_n, n = 1,2,...\}$ in $[T_1,\infty)$ such that

$$\dot{x}(t_n) = 0 \quad and \quad t_n \to \infty \ as \ n \to \infty .$$

It follows from the inequality (3) and condition (O_3) that

$$\int_{T_1}^{\infty}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds \le \frac{C_3+\lambda}{\gamma A} < \infty \ ,$$

and then $\dot{x}(t)x^{-\beta}(t) \in L^2(T_1,\infty)$, which means that there exists a constant $M_0 > 0$ such that

$$\int_{T_1}^{t}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds \le M_0 .$$

Then by Schwarz's inequality we note that

$$\left|\int_{T_1}^{t}\dot{x}(s)/x^{\beta}(s)\, ds\right|^2 \le \left(\int_{T_1}^{t} ds\right)\left(\int_{T_1}^{t}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds\right)$$

$$\le t\int_{T_1}^{t}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds$$

$$\le M_0 t \ ,$$

thus, for all $t \ge T_1$, we have

$$-\sqrt{M_0 t} \le \frac{1}{1-\beta}\left(x^{1-\beta}(t) - x^{1-\beta}(T_1)\right) \le \sqrt{M_0 t} \ .$$

Since $1-\beta = {}^{(1-\gamma)}/_{2} < 0$. Then we obtain

$$x^{1-\beta}(T_1) + (\beta-1)\sqrt{M_0 t} \ge x^{1-\beta}(t) \ge (1-\beta)\sqrt{M_0 t} + x^{1-\beta}(T_1)$$

$$\ge (1-\beta)\sqrt{M_0 t} - x^{1-\beta}(T_1) \ ,$$

which implies that

$$x^{1-\gamma}(t) \le \left(x^{1-\beta}(T_1) + (\beta-1)\sqrt{M_0 t}\right)^2 \qquad (*)$$

On the other hand since $2(\beta-1) = \gamma - 1 \quad and \quad \gamma > 1$ then we have

$$4(\beta-1)^2 = \gamma^2 - 2\gamma + 1 < 2\gamma(\gamma-1)$$

so we obtain

$$\frac{1}{(\beta-1)^2} > \frac{1}{2(\beta-1)^2} > \frac{1}{\gamma(\gamma-1)} \qquad (**)$$

Now, inequalities (*) and (**) give us the following inequality is valid

$$-\frac{x^{-\gamma+1}(t)}{\gamma(\gamma-1)} > -\left[\frac{x^{1-\beta}(T_1) + (\beta-1)\sqrt{M_0 t}}{\beta-1}\right]^2$$

Hence from (6) we obtain , for all $t \ge T_1$, that

$$-\gamma B\left[\frac{x^{1-\beta}(T_1)+(\beta-1)\sqrt{M_0 t}}{\beta-1}\right]^2+\int_{T_1}^{t}\int_{T_1}^{s} q(u)\,du\,ds \le C_3(t-T_1) \qquad (7)$$

Dividing (7) by t and taking the upper limit as $t\to\infty$, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_{T_1}^{t}\int_{T_1}^{s} q(u)\,du\,ds < \infty,$$

which is a contradiction to the condition (O_4). Next, suppose that $\dot{x}(t)>0$ *for* $t\ge T_3\ge T_1$. Thus

$$x(t)\ge x(T_3) \text{ and } x^{-\gamma+1}(t)\le x^{-\gamma+1}(T_3) .$$

We deduce from (6) that

$$\frac{\beta}{(1-\gamma)}x^{1-\gamma}(T_3)+\int_{T_1}^{t}\int_{T_1}^{s} q(u)duds \le C(t-T_1) \qquad (8)$$

Dividing (8) by t and taking the upper limit as $t\to\infty$, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_{T_1}^{t}\int_{T_1}^{s} q(u)\,du\,ds < \infty ,$$

which again contradicts the condition (7) . Finally, we assume that $\dot{x}(t)<0$ *for* $t\ge T_3\ge T_1$.

By condition (O_3), we can estimate (3) as follows,

$$-\frac{r(t)\dot{x}(t)}{x^{\gamma}(t)}\ge -(C_3+\lambda)+\gamma A\int_{T_1}^{t}\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds \qquad (9)$$

Since $r(t)\le B$ *and* $\dot{x}(t)<0$. Then $-\frac{r(t)\dot{x}(t)}{x^{\gamma}(t)}\le -\frac{B\dot{x}(t)}{x^{\gamma}(t)}$, substituting in (9), we obtain

$$-\frac{\dot{x}(t)}{x^{\gamma}(t)}\ge -k+\gamma\frac{A}{B}\int_{T_1}^{t}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds \qquad (10)$$

where

$$k=(C_3+\lambda)/B \text{ and } \gamma\frac{A}{B}\int_{T_1}^{T_3}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds\ge 0 .$$

If the integral in (10) is finite as $t\to\infty$, we can deduce a contradiction as a similar way as the case when $\dot{x}(t)$ is oscillatory. Otherwise we may choose $T\ge T_3$ such that

$$\gamma\frac{A}{B}\int_{T_3}^{T}\frac{\dot{x}^2(s)}{x^{\gamma+1}(s)}ds=1+k \qquad (11)$$

for $t\ge T$, we multiply (10) through by

$$-\dot{x}(t)/x(t)\Big/\left\{-k+\frac{\gamma A}{B}\int_{T_3}^{t}\left(\dot{x}(s)/x^{\beta}(s)\right)^2 ds\right\}$$

and integrating from T to t as follows

$$\int_{T}^{t}\left[\overset{\bullet 2}{x}(s)\Big/x^{\gamma+1}(s)\Bigg/\left\{-k+\frac{\gamma A}{B}\int_{T_3}^{s}\left(\overset{\bullet}{x}(u)\Big/x^{\beta}(u)\right)^{2}du\right\}\right]ds\geq\int_{T}^{t}-\frac{\overset{\bullet}{x}(s)}{x(s)}ds \tag{12}$$

For the integral in L.H.S. of (12), since

$$\frac{d}{ds}\left[-k+\frac{\gamma A}{B}\int_{T_3}^{s}\frac{\overset{\bullet 2}{x}(u)}{x^{\gamma+1}(u)}du\right]=\frac{\gamma A}{B}\times\frac{d}{ds}\left(\int_{T_3}^{s}\frac{\overset{\bullet 2}{x}(u)}{x^{\gamma+1}(u)}du\right)$$

$$=\gamma\frac{A}{B}\times\frac{\overset{\bullet 2}{x}(s)}{x^{\gamma+1}(s)}\times 1$$

Therefore, we obtain that

$$\frac{B}{\gamma A}\int_{T}^{t}\left[\left\{\frac{\gamma A}{B}\overset{\bullet 2}{x}(s)\Big/x^{\gamma+1}(s)\right\}\Bigg/\left\{-k+\frac{\gamma A}{B}\int_{T_3}^{s}\frac{\overset{\bullet 2}{x}(u)}{x^{\gamma+1}(u)}du\right\}\right]ds=\frac{B}{\gamma A}\ln\left\{-k+\frac{\gamma A}{B}\int_{T_3}^{t}\frac{\overset{\bullet 2}{x}(u)}{x^{\gamma+1}(u)}du\right\} \tag{13}$$

Using (13) in the inequality (12), we obtain

$$\ln\left\{-k+\frac{\gamma A}{B}\int_{T_3}^{t}\frac{\overset{\bullet 2}{x}(s)}{x^{\gamma+1}(s)}ds\right\}\geq\frac{\gamma A}{B}\ln\frac{x(T)}{x(t)}=\ln\left[\frac{x(T)}{x(t)}\right]^{\frac{\gamma A}{B}}$$

this together with (10) yields $-\dfrac{\overset{\bullet}{x}(t)}{x^{\gamma}(t)}\geq\dfrac{x^{\frac{\gamma A}{B}}(T)}{x^{\frac{\gamma A}{B}}(t)}$, which implies that $-\overset{\bullet}{x}(t)\geq x^{\frac{\gamma A}{B}}(T)x^{\gamma(1-\frac{A}{B})}(t)$

so, we have

$$\overset{\bullet}{x}(t)\leq -x^{\frac{\gamma A}{B}}(T)x^{\gamma(1-\frac{A}{B})}(t) \tag{14}$$

But we have $0<A\leq B$, this implies that $1-\dfrac{A}{B}\geq 0$. Also, since $x(t)>0$. This means that there exists a constant θ such that $x(t)\geq\theta>0$. Then we have from (14) that

$$\overset{\bullet}{x}(t)\leq -x^{\frac{\gamma A}{B}}(T)\theta^{\gamma(1-\frac{A}{B})}=-M',M'>0\,.$$

Therefore $x(t) \le -M'(t-T_1) + x(T_1)$, which implies that $x(t) \to -\infty \quad as \quad t \to \infty$. But this contradicts the assumption that $x(t) > 0$ .This completes the proof.

Example 1. Consider the following differential equation

$$\left(\left(1+ t/_{(t+1)}\right) \dot{x}(t)\right)^{\bullet} + (t-e^t)|x(t)|^{\gamma} \, sign x(t) = 0 \,, \qquad t \ge t_0 \ge 0, 1 < \gamma \,. \tag{15}$$

Note that

$$(i)\ 1 \le r(t) = 1 + \frac{t}{t+1} < 2 \,,$$

$$(ii)\ \dot{r}(t) = \frac{1}{(t+1)^2} > 0 \quad for\ all\ t \ge 0$$

$$(iii)\ \liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds = \liminf_{t\to\infty} \int_{t_0}^{t} \left[s - e^s\right] ds$$

$$= \liminf_{t\to\infty} \left[\frac{s^2}{2} - e^s\right|_{t_0}^{t}$$

$$= \liminf_{t\to\infty} \left[\frac{t^2}{2} - e^t - \frac{t_0^{\ 2}}{2} + e^{t_0}\right]$$

$$= \infty > -\lambda > -\infty \,; \lambda > 0,$$

$$(iiii)\ \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t}\int_{t_0}^{s} q(u)duds = \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left[\frac{s^2}{2} - e^s - \frac{t_0^{\ 2}}{2} + e^{t_0}\right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{t} [\frac{t^3}{6} - e^t - \frac{t_0^{\ 2}}{2} t + te^{t_0}$$

$$- \frac{t_0^{\ 3}}{6} + e^{t_0} + \frac{t_0^{\ 3}}{2} - t_0 e^{t_0}]$$

$$= \infty \,.$$

Hence, by the Theorem 1, we conclude that the given equation is oscillatory for all $\gamma > 1$

Theorem 2:

Suppose that (O_1) and (O_3), (O_4) hold and suppose in addition that

(O_5) $\dot{r}(t) \leq 0 \quad for \quad t \geq t_0$

Then the equation (1) is oscillatory when $\gamma > 1$.

Proof : Assume the contrary , then there exists a solution $x(t)$ which may be assumed to be positive on $[T_1, \infty)$ for some $T_1 \geq t_0 \geq 0$. As in the proof of Theorem 1 , we obtain the inequality (4) , that is

$$\int_{T_1}^{t} \frac{r(s)\dot{x}(s)}{x^{\gamma}(s)} ds + \gamma A \int_{T_1}^{t}\int_{T_1}^{s} \left(\dot{x}(u) / x^{\beta}(u) \right)^2 du ds + \int_{T_1}^{t}\int_{T_1}^{s} q(u)\, du\, ds \leq C_3(t - T_1) \qquad (4)$$

Now , for the first integral of (4), we have by condition (O_5) that $r(t)$ is non increasing on $[T_1, \infty)$.So , by the Bonnet theorem there exists $a_t \in [T_1, t]$ such that

$$\int_{T_1}^{t} r(s) \frac{\dot{x}(s)}{x^{\gamma}(s)} ds = r(T_1) \int_{T_1}^{a_t} \frac{\dot{x}(s)}{x^{\gamma}(s)} ds = \frac{r(T_1)}{1-\gamma} \left[x^{-\gamma+1}(a_t) - x^{-\gamma+1}(T_1) \right]$$

By substituting in (4), we obtain

$$\frac{r(T_1) x^{1-\gamma}(a_t)}{1-\gamma} + \gamma A \int_{T_1}^{t}\int_{T_1}^{s} \left(\dot{x}(u) / x^{\beta}(u) \right)^2 du ds + \int_{T_1}^{t}\int_{T_1}^{s} q(u)\, du\, ds$$

$$\leq C_3(t - T_1) + \frac{r(T_1) x^{1-\gamma}(T_1)}{1-\gamma}$$

$$\leq C_3(t - T_1) \qquad (16)$$

Since $r(T_1) \leq B$. Then $\frac{r(T_1) x^{1-\gamma}(a_t)}{1-\gamma} \geq \frac{B x^{1-\gamma}(a_t)}{1-\gamma}$. So from (16) we have

$$\frac{B x^{1-\gamma}(a_t)}{1-\gamma} + \gamma A \int_{T_1}^{t}\int_{T_1}^{s} \left(\dot{x}(u) / x^{\beta}(u) \right)^2 du ds + \int_{T_1}^{t}\int_{T_1}^{s} q(u)\, du\, ds \leq C_3(t - T_1)$$

then the rest of the proof is similar that of the proof of Theorem 2 and then it will be omitted.

Example 2

$$\left[\left(e + 1/(t+1) \right) \dot{x}(t) \right]^{\bullet} + \left(\frac{t}{5} + \sin t \right) |x(t)|^{\gamma} signx(t) = 0 \ , \qquad t \geq t_0 \geq 0 \, , 1 < \gamma \qquad (17)$$

Note that

(i) $e < r(t) = e + 1/(t+1) < e + 1$

(ii) $\dot{r}(t) = -\frac{1}{(t+1)^2} < 0 \quad for\, all\, t \geq 0$,

$$(iii)\ \liminf_{t\to\infty}\int_{t_0}^{t} q(s)ds = \liminf_{t\to\infty}\int_{t_0}^{t}\left(\frac{s}{5}+\sin s\right)ds$$

$$= \liminf_{t\to\infty}\left[\frac{t^2}{10}-\cos t-\frac{{t_0}^2}{10}+\cos t_0\right]$$

$$= \infty > -\lambda > -\infty\ ;\lambda > 0,$$

$$(iiii)\ \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)duds = \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left[\frac{s^2}{10}-\cos s-\frac{{t_0}^2}{10}+\cos t_0\right]ds$$

$$= \limsup_{t\to\infty}\frac{1}{t}[\frac{t^3}{30}-\sin t-\frac{{t_0}^2}{10}t+t\cos t_0$$

$$-\frac{{t_0}^3}{30}+\sin t_0+\frac{{t_0}^3}{10}-t_0\cos t_0 = \infty$$

Hence, by the Theorem 2, we conclude that the given equation is oscillatory for all $\gamma > 1$

Remark

Theorems 1 and 2 are extended for the results of Nasr [11], Onose [12] and Wong [15].

3.CONCLUSIONS

In a conclusion, we study the oscillation behavior for some nonlinear differential equations. However, we obtained some sufficient conditions for all the solutions to be oscillate. Our results extend and/or generalize some previous theorems in the literature. Some examples are also given to illustrate the new results.

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